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GURU GOBIND SINGH INDRAPRASTHA UNIVERSITY, DELHI
BACHELOR OF COMMERCE(Hons)

BCOM 106- Business Mathematics

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Unit -I

Permutation and Combination

INTRODUCTION

- Permutation and combination has lately emerged as an important topic for many entrance examinations. This is primary because questions from the topic require analytical skill and a logical bend of mind. Even students who do not have mathematics as a subject can handle them if they have a fairly good understanding of the concepts and their application. Hence anyone who is well-versed in different methods of counting and basic calculations will be able to solve these problems easily.

- IMPORTANT NOTATION

- $n!$ (Read as n factorial)
- Product of first n positive integers is called n factorial
- $n! = 1 \times 2 \times 3 \times 4 \times 5 \times$é n
- $n! = (n$ ī l$)!$ $n \in N$
- In special case $0! = 1$

MEANING OF PERMUTATION AND COMBINATION

Permutation

The arrangement made by taking some or all elements out of a number of things is called a permutation.

The number of permutations of n things taking r at a time is denoted by ${}^nP_r$ and it is defined as under:

Combination

The group or selection made by taking some or all elements out of a number of things is called a combination.

The number of combinations of n things taking r at a time is denoted by ${}^nC_r$ or and it is defined as under:

Here $n!$ = Multiple of n natural number

Some Important Results of Permutations

1. ${}^nP_{n-1} = {}^nP_n$
2. ${}^nP_n = n!$
3. ${}^nP_r = n\,({}^{n-1}P_{r-1})$
4. ${}^nP_r = (n - r + 1) \times {}^nP_{r-1}$
5. ${}^nP_r = {}^{n-1}P_r + r\,({}^{n-1}P_{r-1})$

Types of Permutations

Permutations with Repetition

When you have ***n*** things to choose from ... you have ***n*** choices each time!

When choosing ***r*** of them, the permutations are:

n × n × ... *(r times)*

(In other words, there are **n** possibilities for the first choice, AND THEN there are **n** possibilities for the second choice, and so on, multiplying each time.)

Which is easier to write down using an exponent of **r**?

$$\mathbf{n \times n \times \ldots (r\ times) = n^r}$$

Example: in the lock above, there are 10 numbers to choose from (0,1,..9) and you choose 3 of them:

$$\mathbf{10 \times 10 \times \ldots (3\ times) = 10^3 = 1{,}000\ permutations}$$

Permutations without Repetition

In this case, you have to **reduce** the number of available choices each time.

For example, what order could 16 pool balls be in?

After choosing, say, number "14" you can't choose it again.

So, your first choice would have 16 possibilities, and your next choice would then have 15 possibilities, then 14, 13, etc. And the total permutations would be:

$$16 \times 15 \times 14 \times 13 \times ... = 20,922,789,888,000$$

But maybe you don't want to choose them all, just 3 of them, so that would be only:

$$16 \times 15 \times 14 = 3,360$$

In other words, there are 3,360 different ways that 3 pool balls could be selected out of 16 balls.

But how do we write that mathematically? Answer: we use the "factorial function"

The **factorial function** (symbol :!) just means to multiply a series of descending natural numbers. Examples:

- $4! = 4 \times 3 \times 2 \times 1 = 24$
- $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5,040$
- $1! = 1$

There are also two types of combinations (remember the order does **not** matter now):

1. **Repetition is Allowed**: such as coins in your pocket (5,5,5,10,10)
2. **No Repetition**: such as lottery numbers (2,14,15,27,30,33)

1. Combinations with Repetition

Actually, these are the hardest to explain, so I will come back to this later.

2. Combinations without Repetition

This is how lotteries work. The numbers are drawn one at a time, and if you have the lucky numbers (no matter what order) you win!

The easiest way to explain it is to:

- assume that the order does matter (i.e. permutations),
- then alter it so the order does **not** matter.

Going back to our pool ball example, let us say that you just want to know which 3 pool balls were chosen, not the order.

We already know that 3 out of 16 gave us 3,360 permutations.

But many of those will be the same to us now, because we don't care what order!

For example, let us say balls 1, 2 and 3 were chosen. These are the possibilities:

Order does matter	Order doesn't matter
1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1	1 2 3

So, the permutations will have 6 times as many possibilities.

In fact there is an easy way to work out how many ways "1 2 3" could be placed in order, and we have already talked about it. The answer is:

$$\mathbf{3! = 3 \times 2 \times 1 = 6}$$

(Another example: 4 things can be placed in **4! = 4 × 3 × 2 × 1 = 24** different ways, try it for yourself!)

So, all we need to do is adjust our permutations formula to **reduce it** by how many ways the objects could be in order (because we aren't interested in the order any more):

$$\frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

That formula is so important it is often just written in big parentheses like this:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

where ***n*** is the number of things to choose from, and you choose ***r*** of them
(No repetition, order doesn't matter)

It is often called "n choose r" (such as "16 choose 3")

And is also known as the "Binomial Coefficient"

Notation

As well as the "big parentheses", people also use these notations:

$$C(n,r) = {}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example

So, our pool ball example (now without order) is:

$$\frac{16!}{3!(16-3)!} = \frac{16!}{3!\times 13!} = \frac{20{,}922{,}789{,}888{,}000}{6\times 6{,}227{,}020{,}800} = 560$$

Or you could do it this way:

$$\frac{16\times 15\times 14}{3\times 2\times 1} = \frac{3360}{6} = 560$$

It is interesting to also note how this formula is nice and **symmetrical**:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r} = \binom{n}{n-r}$$

In other words choosing 3 balls out of 16, or choosing 13 balls out of 16 have the same number of combinations.

$$\frac{16!}{3!\,(16-3)!} = \frac{16!}{13!\,(16-13)!} = \frac{16!}{3! \times 13!} = 560$$

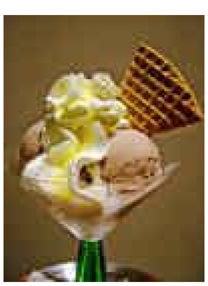
Pascal's Triangle

You can also use Pascal's Triangle to find the values. Go down to row "n" (the top row is 0), and then along "r" places and the value there is your answer. Here is an extract showing row 16:

```
        1    14    91    364  ...
     1    15    105    455    1365  ...
  1    16    120    560    1820   4368  ...
```

1. Combinations with Repetition

OK, now we can tackle this one ...



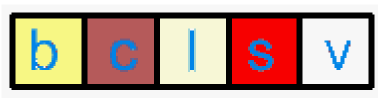
Let us say there are five flavors of ice-cream: **banana, chocolate, lemon, strawberry and vanilla**. You can have three scoops. How many variations will there be?

Let's use letters for the flavors: {b, c, l, s, v}. Example selections would be

- {c, c, c} (3 scoops of chocolate)
- {b, l, v} (one each of banana, lemon and vanilla)
- {b, v, v} (one of banana, two of vanilla)

(And just to be clear: There are **n=5** things to choose from and you choose **r=3** of them.
Order does not matter, and you **can** repeat!)

Now, I can't describe directly to you how to calculate this, but I can show you a **special technique** that lets you work it out.



Think about the ice cream being in boxes, you could say "move past the first box, then take 3 scoops, then move along 3 more boxes to the end" and you will have 3 scoops of chocolate!

So, it is like you are ordering a robot to get your ice cream, but it doesn't change anything, you still get what you want.

Now you could write this down as →○○○→→→(arrow means **move**, circle means **scoop**).

In fact the three examples above would be written like this:

{c, c, c} (3 scoops of chocolate): →○○○→→→

{b, l, v} (one each of banana, lemon and vanilla): ○→→○→→○

{b, v, v} (one of banana, two of vanilla): ○→→→→○○

OK, so instead of worrying about different flavors, we have a ***simpler*** problem to solve: "how many different ways can you arrange arrows and circles"

Notice that there are always 3 circles (3 scoops of ice cream) and 4 arrows (you need to move 4 times to go from the 1st to 5th container).

So (being general here) there are ***r*** + ***(n-1)*** positions, and we want to choose ***r*** of them to have circles.

This is like saying "we have ***r*** + ***(n-1)*** pool balls and want to choose ***r*** of them". In other words it is now like the pool balls problem, but with slightly changed numbers. And you would write it like this:

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

where ***n*** is the number of things to choose
from, and you choose ***r*** of them
(Repetition allowed, order doesn't matter)

Interestingly, we could have looked at the arrows instead of the circles, and we would have then been saying "we have ***r* + *(n-1)*** positions and want to choose ***(n-1)*** of them to have arrows", and the answer would be the same...

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1} = \frac{(n+r-1)!}{r!(n-1)!}$$

So, what about our example, what is the answer?

$$\frac{(5+3-1)!}{3!\,(5-1)!} = \frac{7!}{3!\times 4!} = \frac{5040}{6\times 24} = 35$$

Circular Permutation: - Total number of circular permutation = n(n - 1)!

In Conclusion

Phew, that was a lot to absorb, so maybe you could read it again to be sure!

But knowing *how* these formulas work is only half the battle. Figuring out how to interpret a real world situation can be quite hard.

But at least now you know how to calculate all 4 variations of "Order does/does not matter" and "Repeats are/are not allowed".

ILLUSTRATIONS

If ${}^nC_{10} = {}^nC_{14}$ then find the value of n

Solution

${}^nC_{10} = {}^nC_{14} \Rightarrow n = (10 + 14) = 24 \quad (\because n = p + q)$

If ${}^nC_3 = 220$ then find the value of n.

Solution

$^{n}C_3 = 220$

$\therefore n(n - 1)(n - 2) = 1320$
$\therefore n(n - 1)(n - 2) = 12 \times 11 \times 10$
$\therefore n = 12$

If $^{n}P_5 = 20_nP_3$ then find the value of *n.*

Solution

$^{n}P_5 = 20_nP_3$

$\therefore (n - 3)! = 20(n - 5)!$
$\therefore (n - 3)(n - 4)(n - 5)! = 20(n - 5)!$
$\therefore (n - 3)(n - 4)! = 20$
$\therefore n^2 - 7n + 12 = 20$
$\therefore n^2 - 7n - 8 = 0$
$\therefore (n - 8)(n + 1) = 0$
$\therefore n = 8$ or $n = -1 \notin N$
$\therefore n = 8$

If $^{n}C_r + {}^{n}C_{r+1}, = {}^{n+1}C^{x}$ then find the value of *x.*

Solution

$$^{n}C_r + {}^{n}C_{r+1} = {}^{n+1}C_{r+1} \text{ (formula)}$$

and $^{n}C_r + {}^{n}C_{r+1} = {}^{n}C^{x}$ (given)
By comparing above two statements we can say that
$x = r + 1$

ANALYTICAL EXERCISES

1. If then find the value of *n.*

2. If then find the value of *n.*
3. If then find the value of *x.*
4. If $^{n}P_3 = 60$ then find the value of *n*.
5. If then find the value of *r*.
6. Find the value of $(n + 1)! - n!$
7. If then find the value of *n.*
8. If $^{5}P_r = 60$ then find *r.*
9. If $^{18}C = {}^{18}C_{r+2}$ then find *r.*
10. Find the value of $^{n-2}C_r + 2\ {}^{n-2}C_{r-1} + {}^{n-2}C_{r-2}$.
11. Find the value of $^{12}C_3 + 2\ {}^{12}C_4 + {}^{12}C_5$.
12. Find the value of $^{n}C^{r} + {}^{n}C_{r-1}$.
13. Find the value of $^{n-1}P^{r} + r.\ {}^{n-1}P_{r-1}$.
14. Find the value of
15. Find the value of $^{12}C_4 + {}^{12}C_3$.
16. If $^{499}C_{92} + {}^{n}C_{91} = {}^{500}C_{92}$ then find the value of *n*
17. If then find the value of *n.*
18. If $^{7}P_n + {}^{7}P_{n-3} = 60$ then find the value of *n*.
19. If 720 then find the value of *n.*
20. If $^{10}P_r = 720$ then find the value of *r*.
21. A man has 6 friends to invite. In how many ways can he send invitation cards to them if he has three servants to carry the cards?

Principle of Mathematical Induction:-

INTRODUCTION

Consider the set *N* of natural numbers. *N* has two characteristic proper ties.

1. *N* contains the natural number 1.
2. *N* is closed with respect to addition of 1 to each of its numbers.

Therefore to determine whether a set *k* consisting of natural numbers is the set of all natural numbers, we have to verify the following two conditions on *K:*

1. Does $1 \in K$?
2. For each natural number $K \in K$; is it true that $K \in K$?

When answer to both the questions is ñyesò then K is N. It gives several important principles for establishing the truth of certain classes of statements.

HISTORICAL INTRODUCTION

The discovery of the principle of mathematical induction is generally attributed to the French mathematician Blaise Pascal (1623ï 1662). However, the principal has been used by the Italian mathematician Francesco Maurolycus (1445ï 1575) in his writings. The writings of Bhaska racharya (1150 A.D.) also lead us to believe that he knew of this principle.

The first one to use the term ñinductionò was the English mathematician John Walls (1616ï 1703). Later on the Swiss mathematician James Bar noul i (1655ï 1705) used the principle to provide a proof of the binomial theorem about which we shall learn later in this book.

The term *mathematical induction* in the modern sense was used by English mathematician Augustus Deò Morgan (1806ï 1871) in his article ñinductionò (Mathematics) in Penny Cyclopaedia, London 1938. The term gained immediate acceptance by the mathematical community and during the next fifty years or so it was universally accepted.

PRINCIPLE OF FINITE INDUCTION (PFI)

If we denote the given statement or formula by $P(n)$, for all positive integral values of n, then the proof of this statement with the help of the principle of mathematical induction consists of three steps.

Step1: Verify that the result is tr ue for the first available value of n, generally for $n = 1$, i.e. verify that $p(1)$ is true.

Step2: Assume that the result is tr ue for a positive integral value K of n,. i.e. assume that $p(k)$ is true.

Step3: Now using the result that $p(k)$ is true, prove that the result is also tr ue for $n = k + 1$, i.e. prove that $p(k + 1)$ is also true.

Having followed the above three steps it can be said that the result is proved with the help of principle of mathematical induction.

Q:- For positive integer value of n, prove that $1 + 3 + 5 + 7 + é \quad (2n ï \ 1) = n^2$

Solution

Here $p(n) = 1 + 3 + 5 + 7 + é \quad (2n ï \ 1) = n^2$

Let us verify the given result for $n = 1$

L.H.S. = 1 R.H.S. = $(1)^2 = 1$

∴ L.H.S. = R.H.S.

Now we shall assume that the result is true for $n = k$

i.e. we shall prove that $p(k)$ is true

$\therefore p(k) = 1 + 3 + 5 + 7 + \ldots (2k - 1) = k^2$

Now we shall prove that the result is true for $n = k + 1$

i.e. we shall prove that

$1 + 3 + 5 + 7 + \ldots (2k - 1) + (2k + 1) = (k + 1)^2$

L.H.S. $= 1 + 3 + 5 + 7 + \ldots (2k - 1) + (2k + 1)$

$= [1 + 3 + 5 + 7 + \ldots (2k - 1)] + (2k + 1)$

$= k^2 + 2k + 1$

$= (k + 1)$

Thus it is proved that the result is also true for $n = k + 1$ i.e. if $p(k)$ is true, $p(k + 1)$ is also true.

Sequence and Series

INTRODUCTION

Sequence and series is a mathematical concept that draws majorly from the basic number system and the simple concepts of arithmetic. This is the reason that makes it an important topic for this exam. On an average 2-3 questions have been asked from the topic almost every year. This topic is important for other exams also for example, CAT, IIFT, SNAP, XAT, MAT and JMET. The application of logic or some very simple concepts of algebraic calculations can be solved simply.

SEQUENCE AND SERIES

Let us consider the following progressions:

1, 3, 5, 7, 9 …

and 2, 6, 8, 12 …

It can be observed here that each of these two series shares some or the other common properties.

If the terms of a sequence are written under same specific conditions then the sequence is called a *progression.*

With respect to preparation for the BBA, we will confine ourselves only to the following standard series of progression:

1. Arithmetic progression
2. Harmonic progression
3. Geometric progression

ARITHMETIC PROGRESSION

Definition

If in any progression consecutive difference between any two terms is same, then that progression is said to be an *arithmetic progression* (A.P.), e.g.

a, a + d, a + 2d, a + 3d… a + (n −1)d i.e. *Tn = a* + (n ī l) *d. Tn* is the last term.

Note that d = T_2 ī T_1 = T_3 ī T_2 = T_4 ī T_3 = é

Sum of an A.P.

If S_n is the sum of first n terms of the A.P.

a, a + d, a + 2d é

i.e. If S_n = a + (a + *d)* + (a + *2d) + …up to n terms*

then

Arithmetic Mean

If three numbers *a, b, c* are in A.P. then *b* is called the arithmetic mean between *a* and *c*.

1. The arithmetic mean between two numbers *a* and *b* is .
2. A_1, A_2 … A_n are said to be *n* arithmetic means between two numbers *a* and *b*. *A, b are* in A.P. and if *d* is the common difference of this A.P,

GEOMETRIC PROGRESSION

If in any sequence, consecutive ratio between any two terms is same it is said to be a *geometric progression* (G.P).

e.g. $a, ar^2, ar^3 \ldots ar_{n-1}$

∴ $T_n = ar_{n-1}$

Where a = first term, r = common ratio & n = number of term.

Sum of a G.P

If S_n is the sum of first n terms of G.P. $a, ar^2, ar^3 \ldots ar^{n-1}$
i.e S_n = a + ar + ar^2 + é + ar^{n-1}
  s$_n$ = a(r^n - i) /r ï 1 where r> 1
S$_n$ = a(1 - r^n) /1- r where r< 1
We can take the above S_n formula for finite progression and s_n = ar + ar + ar^2 + é up to infinity.
S$_n$ = a/(1 - r) where -1< r < 1

Geometric Mean

If three non-zero numbers a, b, c are in G.P. then b is called the geometric mean between a and a *and b*
 i.e. a / b = b / c
since , b^2 = ac

DO YOU KNOW?

1. Three non-zero numbers in *a, b, c* are in G.P. if $b^2 = ac.$
2. Three non-zero numbers in *a, b, c* are in G.P. if .
3. *If A, G, H* denote respectively the A.M., G.M., H.M. between two distinct positive numbers then
    1. *A, G, H* are in G.P.
    2. *A > G > H.*
4. In a G.P. the product of terms equidistant from the beginning and end is constant.
5. A sequence (or a series) is both an A.P. as well as a G.P. if it is a constant sequence i.e. if all the terms are equal.

Important Notes on A.P. and G.P.

It is convenient to take

1. Three numbers in A.P. as $a - d, a, a + d$

2. Four numbers in A.P. as $a - 3d, a - d, a + d, a + 3d$
3. Three numbers in G.P. as a / r, a, ar
4. Five numbers in G.P. as a /r^2 , a/ r , a, ar, ar^2

UNIT –II

Linear Algebra

INTRODUCTION: MATRIX / MATRICES

1. A rectangular array of m×n numbers arranged in the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called an m×n *matrix*.

e.g. $\begin{bmatrix} 2 & 3 & 4 \\ 1 & -8 & 5 \end{bmatrix}$ is a 2×3 matrix.

e.g. $\begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ is a 3×1 matrix.

2. If a matrix has m *rows* and n *columns*, it is said to be *order* m×n.

e.g. $\begin{bmatrix} 2 & 0 & 3 & 6 \\ 3 & 4 & 7 & 0 \\ 1 & 9 & 2 & 5 \end{bmatrix}$ is a matrix of order 3×4.

e.g. $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 5 \\ -1 & 3 & 0 \end{bmatrix}$ is a matrix of order 3.

3. $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ is called a *row matrix* or *row vector*.

4. $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is called a *column matrix* or *column vector*.

e.g. $\begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ is a column vector of order 3×1.

e.g. $\begin{bmatrix} -2 & -3 & -4 \end{bmatrix}$ is a row vector of order 1×3.

5. If all elements are real, the matrix is called a real matrix.

6. $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ is called a *square matrix* of order n.

And $a_{11}, a_{22}, \ldots, a_{nn}$ is called *the principal diagonal.*

e.g. $\begin{bmatrix} 3 & 9 \\ 0 & -2 \end{bmatrix}$ is a square matrix of order 2.

7. Notation : $[a_{ij}]_{m\times n}$, $(a_{ij})_{m\times n}$, A , ...

SOME SPECIAL MATRIX.

Def If all the elements are zero, the matrix is called a *zero matrix* or null matrix, denoted by $O_{m\times n}$.

e.g. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 2×2 zero matrix, and denoted by O_2.

Def. Let $A = \left[a_{ij}\right]_{n\times n}$ be a square matrix.

(i) If $a_{ij} = 0$ for all i, j, then A is called a zero matrix.

(ii) If $a_{ij} = 0$ for all i<j, then A is called a *lower triangular matrix*.

(iii) If $a_{ij} = 0$ for all i>j, then A is called a *upper triangular matrix*.

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & & \vdots \\ \vdots & & & & 0 \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ 0 & 0 & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

i.e. Lower triangular matrix Upper triangular matrix

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 4 \end{bmatrix}$ is a lower triangular matrix.

e.g. $\begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix}$ is an upper triangular matrix.

Def. Let $A = \left[a_{ij}\right]_{n\times n}$ be a square matrix. If $a_{ij} = 0$ for all $i \neq j$, then A is called a *diagonal matrix*.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix.

Def. If A is a diagonal matrix and $a_{11} = a_{22} = \cdots = a_{nn} = 1$, then A is called an *identity matrix* or a *unit matrix*, denoted by I_n.

e.g. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ARITHMETICS OF MATRICES.

Def. Two matrices A and B are equal iff they are of the *same order* and their *corresponding elements are equal*.

i.e. $\left[a_{ij}\right]_{m\times n} = \left[b_{ij}\right]_{m\times n} \Leftrightarrow a_{ij} = b_{ij}$ for all i, j.

e.g. $\begin{bmatrix} a & 2 \\ 4 & b \end{bmatrix} = \begin{bmatrix} -1 & c \\ d & 1 \end{bmatrix} \Leftrightarrow a = -1,\ b = 1,\ c = 2,\ d = 4$.

N.B. $\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}$

Def. Let $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{m\times n}$.

Define $A+B$ as the matrix $C=\left[c_{ij}\right]_{m\times n}$ of the same order such that

$c_{ij}=a_{ij}+b_{ij}$ for all i=1,2,...,m and j=1,2,...,n.

e.g. $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}+\begin{bmatrix} 2 & -4 & 3 \\ 2 & -1 & 5 \end{bmatrix}=$

N.B. 1. $\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}+\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}$ is not defined.

2. $\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}+5$ is not defined.

Def. Let $A=\left[a_{ij}\right]_{m\times n}$. Then $-A=\left[-a_{ij}\right]_{m\times n}$ and A-B=A+(-B)

e.g.1 If $A=\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 4 & 0 \\ 3 & -1 & 1 \end{bmatrix}$. Find -A and A-B.

Properties of Matrix Addition.

Let A, B, C be matrices of the same order and O be the zero matrix of the same order. Then

(a) A+B=B+A

(b) (A+B)+C=A+(B+C)

(c) A+(-A)=(-A)+A=O

(d) A+O=O+A

Def. Scalar Multiplication.

Let $A=\left[a_{ij}\right]_{m\times n}$, k is scalar. Then kA is the matrix $C=\left[c_{ij}\right]_{m\times n}$ defined by $c_{ij}=ka_{ij}$, $\forall$i, j.

i.e. $kA=\left[ka_{ij}\right]_{m\times n}$

e.g. If $A=\begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}$,

then -2A= ; $\frac{3}{2}A=$

N.B. (1) -A=(-1)A

(2) A-B=A+(-1)B

Thm. Properties of Scalar Multiplication.

Let A, B be matrices of the same order and h, k be two scalars.

Then (a) k(A+B)=kA+kB

(b) (k+h)A=kA+hA

(c) (hk)A=h(kA)=k(hA)

Def. Let $A=\left[a_{ij}\right]_{m\times n}$. The *transpose* of A, denoted by A^T, or A' , is defined by

$$A^T=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nm} \end{bmatrix}_{n\times m}$$

e.g. $A=\begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}$, then $A^T=$

e.g. $A=\begin{bmatrix} 3 & 0 & -2 \\ 4 & -6 & 1 \end{bmatrix}$, then $A^T=$

e.g. $A=\left[5\right]$, then $A^T=$

N.B. (1) $I^T=$

(2) $A=\left[a_{ij}\right]_{m\times n}$, then $A^T=$

Properties of Transpose.

Let A, B be two m×n matrices and k be a scalar, then

(a) $(A^T)^T =$

(b) $(A+B)^T =$

(c) $(kA)^T =$

Def.8.11 A square matrix A is called a *symmetric matrix* iff $A^T = A$.

i.e. A is symmetric matrix

$\Leftrightarrow A^T = A \Leftrightarrow a_{ij} = a_{ji} \quad \forall$i, j

e.g. $\begin{bmatrix} 1 & 3 & -1 \\ 3 & -3 & 0 \\ -1 & 0 & 6 \end{bmatrix}$ is a symmetric matrix.

e.g. $\begin{bmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ -1 & 3 & 6 \end{bmatrix}$ is not a symmetric matrix.

Def.8.12 A square matrix A is called a *skew-symmetric matrix* iff $A^T = -A$.

i.e. A is skew-symmetric matrix

$\Leftrightarrow A^T = -A \Leftrightarrow a_{ij} = -a_{ji} \quad \forall$i, j e.g.2 Prove that $A = \begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & 5 \\ 1 & -5 & 0 \end{bmatrix}$ is a skew-symmetric matrix.

e.g.3 Is $a_{ii} = 0$ for all i=1,2,...,n for a skew-symmetric matrix?

Matrix Multiplication.

Let $A = \left[a_{ik}\right]_{m\times n}$ and $B = \left[b_{kj}\right]_{n\times p}$. Then the product AB is defined as the m×p matrix $C = \left[c_{ij}\right]_{m\times p}$ where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} \; .$$

i.e. $$AB = \left[\sum_{k=1}^{n} a_{ik}b_{kj}\right]_{m\times p}$$

e.g. Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}_{2\times 3}$. Find AB and BA.

e.g. Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}_{2\times 2}$. Find AB. Is BA well defined?

In general, AB $\neq$ BA.

i.e. matrix multiplication is not commutative.

Properties of Matrix Multiplication.

(a) (AB)C = A(BC)

(b) A(B+C) = AB+AC

(c) (A+B)C = AC+BC

(d) AO = OA = O

(e) IA = AI = A

(f) k(AB) = (kA)B = A(kB)

(g) $(AB)^T = B^T A^T$.

(1) Since AB $\neq$ BA ;

Hence, A(B+C) $\neq$ (B+C)A and A(kB) $\neq$ (kB)A.

(2) $A^2 + kA = A(A + kI) = (A + kI)A$.

(3) $AB - AC = O \Rightarrow A(B - C) = O$

$$\not\Rightarrow A = O \text{ or } B - C = O$$

e.g. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Then $AB - AC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

But $A \neq O$ and $B \neq C$,

so $AB - AC = O \not\Rightarrow A = O$ or $B = C$.

Def. ***Powers of matrices***

For any square matrix A and any positive integer n, the symbol

A^n denotes $\underbrace{A \cdot A \cdot A \cdots A}_{n \text{ factors}}$.

(1) $(A+B)^2 = (A+B)(A+B)$

$$= AA + AB + BA + BB$$

$$= A^2 + AB + BA + B^2$$

(2) If $AB = BA$, then $(A+B)^2 = A^2 + 2AB + B^2$

e.g.6 Let $A = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 & 0 \\ 3 & -1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Evaluate the following :

(a) $(A+2B)C$ (b) $(AC)^2$

(c) $(B^T - 3C)^T D$ (d) $(-2A)^T B - DD^T$

e.g. (a) Find a 2x2 matrix A such that

$$2A-3\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}=\frac{1}{2}\left[A+\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\right].$$

(b) Find a 2x2 matrix $A=\begin{pmatrix}2 & \alpha\\ \beta & \gamma\end{pmatrix}$ such that

$A^T=A$ and $\begin{pmatrix}2 & 1\\ 3 & 0\end{pmatrix}A=A\begin{pmatrix}2 & 1\\ 3 & 0\end{pmatrix}$.

(c) If $\begin{pmatrix}3 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ x\end{pmatrix}=\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}1\\ x\end{pmatrix}$, find the values of x and λ.

e.g.8 Let $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$. Prove by mathematical induction that

$$A^n=\begin{pmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{pmatrix} \quad \text{for } n=1,2,\cdots.$$

e.g.9 (a) Let $A=\begin{pmatrix}a & 1\\ 0 & b\end{pmatrix}$ where $a,b\in R$ and $a\neq b$.

Prove that $A^n=\begin{pmatrix}a^n & \dfrac{a^n-b^n}{a-b}\\ 0 & b^n\end{pmatrix}$ for all positive integers n.

(b) Hence, or otherwise, evaluate $\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}^{95}$.

e.g. (a) Let $A=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$ and B be a square matrix of order 3. Show that if A and B are commutative, then B is a triangular matrix.

(b) Let A be a square matrix of order 3. If for any $x, y, z \in R$, there exists $\lambda \in R$ such that $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, show that A is a diagonal matrix.

(c) If A is a symmetric matrix of order 3 and A is nilpotent of order 2 (i.e. $A^2 = O$), then A=O, where O is the zero matrix of order 3.

Properties of power of matrices :

(1) Let A be a square matrix, then $(A^n)^T = (A^T)^n$.

(2) If $AB = BA$, then

(a) $(A+B)^n = A^n + C_1^n A^{n-1}B + C_2^n A^{n-2}B^2 + C_3^n A^{n-3}B^3 + \cdots + C_{n-1}^n AB^{n-1} + B^n$

(b) $(AB)^n = A^n B^n$.

(3) $(A+I)^n = A^n + C_1^n A^{n-1} + C_2^n A^{n-2} + C_3^n A^{n-3} + \cdots + C_{n-1}^n A + C_n^n I$

e.g.11 (a) Let X and Y be two square matrices such that XY = YX.

Prove that (i) $(X+Y)^2 = X^2 + 2XY + Y^2$

(ii) $(X+Y)^n = \sum_{r=0}^{n} C_r^n X^{n-r} Y^r$ for n = 3, 4, 5, ...

(Note: For any square matrix A , define $A^0 = I$.)

(b) By using (a)(ii) and considering $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, or otherwise, find

$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}^{100}$.

(4 marks)

(c) If X and Y are square matrices,

(i) prove that $(X+Y)^2 = X^2 + 2XY + Y^2$ implies XY = YX ;

(ii) prove that $(X+Y)^3 = X^3 + 3X^2Y + 3XY^2 + Y^3$ does NOT implies XY = YX .

(Hint : Consider a particular X and Y, e.g. $X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$.)

INVERSE OF A SQUARE MATRIX

If a, b, c are real numbers such that ab=c and b is non-zero, then

$a = \frac{c}{b} = cb^{-1}$ and b^{-1} is usually called the multiplicative inverse of b.

(2) If B, C are matrices, then $\frac{C}{B}$ is undefined.

Def. A square matrix A of order n is said to be non-singular or invertible if and only if there exists a square matrix B such that AB = BA = I.

The matrix B is called the multiplicative inverse of A, denoted by A^{-1}

i.e. $AA^{-1} = A^{-1}A = I$.

e.g. Let $A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$, show that the inverse of A is $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$.

i.e. $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$.

e.g.13 Is $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$?

Def. If a square matrix A has an inverse, A is said to be *non-singular* or *invertible*. Otherwise, it is called *singular* or *non-invertible*.

e.g. $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ are both non-singular.

i.e. A is non-singular iff A^{-1} exists.

The inverse of a non-singular matrix is unique.

$I^{-1} = I$, so I is always non-singular.

(2) OA = O $\neq$ I , so O is always singular.

(3) Since AB = I implies BA = I.

Hence proof of either AB = I or BA = I is enough to assert that B is the inverse of A.

e.g.14 Let $A = \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$.

(a) Show that $I - 6A + A^2 = O$.

(b) Show that A is non-singular and find the inverse of A.

(c) Find a matrix X such that $AX = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.

Properties of Inverses

Thm. Let A, B be two non-singular matrices of the same order and λ be a scalar.

(a) $(A^{-1})^{-1} = A$.

(b) A^T is a non-singular and $(A^T)^{-1} = (A^{-1})^T$.

(c) A^n is a non-singular and $(A^n)^{-1} = (A^{-1})^n$.

(d) λA is a non-singular and $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$.

(e) AB is a non-singular and $(AB)^{-1} = B^{-1} A^{-1}$

DETERMINANTS

Def. Let $A=\left[a_{ij}\right]$ be a square matrix of order n. The determinant of A, detA or |A| is defined as follows:

(a) If n=2, $\det A=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$

(b) If n=3, $\det A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

or $\det A=a_{11}a_{22}a_{33}+a_{21}a_{32}a_{13}+a_{31}a_{12}a_{23}$

$-a_{31}a_{22}a_{13}-a_{32}a_{23}a_{11}-a_{33}a_{21}a_{12}$

e.g. Evaluate (a) $\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix}$ (b) $\det\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 1 & -2 & -1 \end{bmatrix}$

e.g. If $\begin{vmatrix} 3 & 2 & x \\ 8 & x & 1 \\ 3 & -2 & 0 \end{vmatrix}=0$, find the value(s) of x.

$$\det A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or $$=-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}-a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

or . . . .

By using $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

e.g. Evaluate (a) $\begin{vmatrix} 3 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 2 & 0 \\ 8 & -2 & 1 \\ 3 & 2 & 3 \end{vmatrix}$

PROPERTIES OF DETERMINANTS

(1) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ i.e. $\det(A^T) = \det A$.

(2) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}$

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix}$

(3) $\begin{vmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & 0 & c_3 \end{vmatrix} = 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 0 \end{vmatrix}$

(4) $$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(5) If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$, then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(6) $$\begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}$$

(7) $$\begin{vmatrix} pa_1 & b_1 & c_1 \\ pa_2 & b_2 & c_2 \\ pa_3 & b_3 & c_3 \end{vmatrix} = p\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ pa_2 & pb_2 & pc_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} pa_1 & pb_1 & pc_1 \\ pa_2 & pb_2 & pc_2 \\ pa_3 & pb_3 & pc_3 \end{vmatrix} = p^3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{pmatrix} pa_1 & pb_1 & pc_1 \\ pa_2 & pb_2 & pc_2 \\ pa_3 & pb_3 & pc_3 \end{pmatrix} = p\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

If the order of A is n, then $\det(\lambda A) = \lambda^n \det(A)$

(8) $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\lambda b_1 & b_1 & c_1 \\ a_2+\lambda b_2 & b_2 & c_2 \\ a_3+\lambda b_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \xlongequal{\alpha C_2 + \beta C_3 + C_1} \begin{vmatrix} x_1+\alpha y_1+\beta z_1 & y_1 & z_1 \\ x_2+\alpha y_2+\beta z_2 & y_2 & z_2 \\ x_3+\alpha y_3+\beta z_3 & y_3 & z_3 \end{vmatrix}$$

e.g. Evaluate (a) $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix}$, (b) $\begin{vmatrix} 5 & 3 & 7 \\ 3 & 7 & 5 \\ 7 & 2 & 6 \end{vmatrix}$

e.g. Evaluate $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

e.g. Factorize the determinant

$$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

e.g. Factorize each of the following :

(a) $\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 2a^3 & 2b^3 & 2c^3 \\ a^2 & b^2 & c^2 \\ 1-a^3 & 1-b^3 & 1-c^3 \end{vmatrix}$

Def. Multiplication of Determinants.

Let $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $|B| = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$

Then $|A||B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$

$$= \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix}$$

Properties :

(1) det(AB)=(detA)(detB) i.e. $|AB| = |A||B|$

(2) |A|(|B||C|)=(|A||B|)|C| A(BC)=(AB)C

(3) |A||B|=|B||A| AB≠BA in general

(4) |A|(|B|+|C|)=|A||B|+|A||C| A(B+C)=AB+AC

e.g. Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Minors and Cofactors

Def. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then A_{ij}, the *cofactor* of a_{ij}, is defined by

$$A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\ A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},\ \ldots,\ A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Since $|A| = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

$$= +a_{21}A_{21} - a_{22}A_{22} + a_{23}A_{23}$$

(a) $$a_{i1}A_{j1} + a_{i2}A_{j2} + a_{i3}A_{j3} = \begin{cases} \det A & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(b) $$a_{1i}A_{1j} + a_{2i}A_{2j} + a_{3i}A_{3j} = \begin{cases} \det A & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

e.g. $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = \det A$, $\quad a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$, etc.

e.g.23 Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and c_{ij} be the cofactor of a_{ij}, where $1 \le i, j \le 3$.

(a) Prove that $A\begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} = (\det A)I$

(b) Hence, deduce that $\begin{vmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{vmatrix} = (\det A)^2$

INVERSE OF SQUARE MATRIX BY DETERMINANTS

Def. The *cofactor matrix* of A is defined as $cofA = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$.

Def. The *adjoint matrix* of A is defined as

$$adjA = (cofA)^T = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}.$$

e.g. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find adjA.

e.g. (a) Let $A = \begin{pmatrix} 1 & 1 & -3 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, find adjA.

(b) Let $B = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & -1 \\ 5 & 1 & -1 \end{pmatrix}$, find adjB.

For any square matrix A of order n ,

A(adjA) = (adjA)A = (detA)

Let A be a square matrix. If det$A \neq 0$, then A is non-singular

and $$A^{-1} = \frac{1}{\det A}(adjA).$$

Proof Let the order of A be n , from the above theorem , $\frac{1}{\det A}(AadjA) = I$

e.g. Given that $A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & -1 \\ 5 & 1 & -1 \end{pmatrix}$, find A^{-1}.

e.g. Suppose that the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-singular , find A^{-1}.

e.g. Given that $A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$, find A^{-1}.

Thm. A square matrix A is non-singular iff det$A \neq 0$.

e.g. Show that $A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ is non-singular.

e.g. Let $A = \begin{pmatrix} x+1 & 2 & x-1 \\ x-1 & 2 & -1 \\ 5 & 7 & -x \end{pmatrix}$, where $x \in R$.

(a) Find the value(s) of x such that A is non-singular.

(b) If x=3 , find A^{-1}.

A is singular (non-invertible) iff A^{-1} does not exist.

Thm. A square matrix A is singular iff detA = 0.

Properties of Inverse matrix.

Let A, B be two non-singular matrices of the same order and λ be a scalar.

(1) $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$

(2) $(A^{-1})^{-1} = A$

(3) $(A^{T})^{-1} = \left(A^{-1}\right)^{T}$

(4) $(A^{n})^{-1} = \left(A^{-1}\right)^{n}$ for any positive integer n.

(5) $(AB)^{-1} = B^{-1}A^{-1}$

(6) The inverse of a matrix is unique.

(7) $\det(A^{-1}) = \frac{1}{\det A}$

N.B. $XY = 0 \not\Rightarrow X = 0 \text{ or } Y = 0$

(8) If A is non-singular , then $AX = 0 \Rightarrow A^{-1}AX = A0 = 0$

$$\Rightarrow X = 0$$

N.B. $XY = XZ \not\Rightarrow X = 0 \text{ or } Y = Z$

(9) If A is non-singular , then $AX = AY \Rightarrow A^{-1}AX = A^{-1}AY$

$$\Rightarrow X = Y$$

(10) $(A^{-1}MA)^{n} = (A^{-1}MA)(A^{-1}MA)\cdots(A^{-1}MA) \quad = A^{-1}M^{n}A$

(11) If $M = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, then $M^{-1} = \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}$.

(12) If $M = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, then $M^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$ where n $\neq 0$.

e.g.31 Let $A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -3 & -1 \\ 0 & 13 & 4 \\ 0 & -33 & -10 \end{pmatrix}$ and $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(a) Find A^{-1} and M^5.

(b) Show that $ABA^{-1} = M$.

(c) Hence, evaluate B^5.

e.g. Let $A = \begin{pmatrix} 3 & 8 \\ 1 & 5 \end{pmatrix}$ and $P = \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}$.

(a) Find $P^{-1}AP$.

(b) Find A^n, where n is a positive integer

e.g. (a) Show that if A is a 3x3 matrix such that $A' = -A$, then detA=0.

(b) Given that $B = \begin{pmatrix} 1 & -2 & 74 \\ 2 & 1 & -67 \\ -74 & 67 & 1 \end{pmatrix}$,

use (a) , or otherwise , to show $\det(I - B) = 0$.

Hence deduce that $\det(I - B^4) = 0$.

e.g. (a) If α , β and γ are the roots of $x^3 + px + q = 0$, find a cubic equation whose

roots are α^2 , β^2 and γ^2.

(b) Solve the equation $\begin{vmatrix} x & 2 & 3 \\ 2 & x & 3 \\ 2 & 3 & x \end{vmatrix} = 0$.

Hence, or otherwise, solve the equation

$x^3 - 38x^2 + 361x - 900 = 0$. e.g. Let M be the set of all 2x2 matrices.

For any $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M$,

define $tr(A) = a_{11} + a_{22}$.

(a) Show that for any A, B, C $\in$ M and $\alpha, \beta \in$ R,

(i) $tr(\alpha A + \beta B) = \alpha tr(A) + \beta tr(B)$,

(ii) $tr(AB) = tr(BA)$,

(iii) the equality "$tr(ABC) = tr(BAC)$" is not necessary true.

(b) Let A $\in$ M.

(i) Show that $A^2 - tr(A)A = -(\det A)I$,

where I is the 2x2 identity matrix.

(ii) If $tr(A^2) = 0$ and $tr(A) = 0$, use (a) and (b)(i) to show that

A is singular and $A^2 = 0$.

(c) Let S, T $\in$ M such that $(ST - TS)S = S(ST - TS)$.

Using (a) and (b) or otherwise, show that

$$(ST - TS)^2 = 0$$

e.g. *Eigen value and Eigenvector*

Let $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ and let x denote a 2x1 matrix.

(a) Find the two real values λ_1 and λ_2 of λ with $\lambda_1 > \lambda_2$

such that the matrix equation

(*) $Ax = \lambda x$

has non-zero solutions.

(b) Let x_1 and x_2 be non-zero solutions of (*) corresponding to

λ_1 and λ_2 respectively. Show that if

$x_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$ and $x_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$

then the matrix $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is non-singular.

(c) Using (a) and (b), show that $AX = X\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

and hence $A^n = X\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}X^{-1}$ where n is a positive integer.

Evaluate $\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}^n$.

UNIT-III

Differential Calculus

INTRODUCTION

Differentiation is one of the most important fundamental operations in calculus. Its theory and preliminary idea is basically dependent upon the concept of limit and continuity. Derivative is to express the rate of change in any function. Derivative means small change in the dependent variable with respect to small change in independent variable. Thus, we can say that derivative is the process of finding the derivative of a continuous function. Derivative is a branch of calculus and its fundamentals and their applications are widely used in mathematics, statistics, economics and financial mathematics.

Properties of Limits (Limit Laws)

Assuming all the limits on the right hand exist:

1. The limit of a constant times a function is the constant times the limit of the function.

   If *b* is a constant, then $\lim_{x \to c}\left(bf(x)\right) = b\left(\lim_{x \to c} f(x)\right)$.

2. The limit of a sum is the sum of the limits. *Also works for differences.

   $$\lim_{x \to c}\left(f(x) + g(x)\right) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

   $$\lim_{x \to c}\left(f(x) - g(x)\right) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$$

3. The limit of a product is the product of the limits.

   $$\lim_{x \to c}\left(f(x)g(x)\right) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right)$$

4. The limit of a quotient is the quotient of the limits (provided that the limit of the

denominator is not zero.)

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}, \text{ provided } \lim_{x \to c} g(x) \neq 0.$$

5. The limit of a constant is the constant.
   For any constant k, $\lim_{x \to c} k = k$.

6. The limit of x as x approaches c is c.
   $\lim_{x \to c} x = c.$

Use the Limit Laws and the graphs of *f* and *g* in Figure 1 to evaluate the following limits:

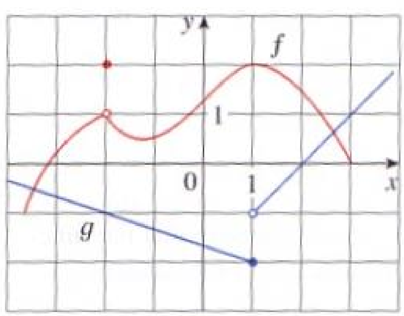


FIGURE 1

a) $\lim_{x \to -2} [f(x) + 5g(x)]$

b) $\lim_{x \to 1} [f(x)g(x)]$

c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$

<u>Continuity</u>

Continuity can fail in the following ways:

- The limit fails to exist. In some texts, this is called an *essential* discontinuity. Any of the examples in the section on limits apply here.
- The limit exists, but the function isn't defined at the point. $y = \frac{\sin x}{x}$ at $x = 0$ is an example.
- The limit exists and the function is defined at the point, but the function output is different from the limit. The function $f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 2 & \text{for } x = 0 \end{cases}$ is an example.

The latter two cases (where the limit exists as x approaches the point in question) are called *removable* discontinuities.

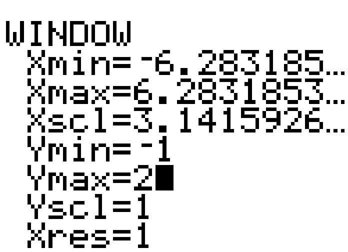
To understand these last three points we need to start taking a look at the concept of *limit* more precisely. What does it really mean when we say that a function f is continuous at $x = c$ if the values of $f(x)$ approach $f(c)$ as x approaches c? What does it mean to approach c? How close to c does x to get?

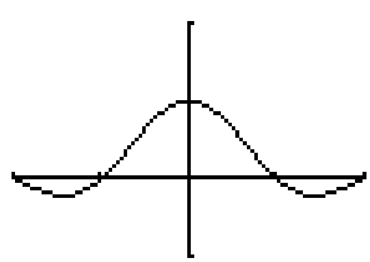
The concept of *limit* is the underpinning of calculus.

The informal definition or notation is $\lim\limits_{x \to c} = L$ if the values of $f(x)$ approach L as x approaches c.

We will look for trends in the values of $f(x)$ as x gets closer to c but $x \neq c$. _____

Example 1: $\lim\limits_{\theta \to 0}\left(\frac{\sin\theta}{\theta}\right)$ (Use radians.)





It appears from the graph that as θ approaches 0 from either side that the value of $\frac{\sin\theta}{\theta}$ appears to approach __________. The actual value of $\frac{\sin\theta}{\theta}$ when θ = 0 is __________.

Therefore the limits exists but the function is not continuous at θ = 0.

While it appears that θ approaches 0 from either side that the value of $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ we are still very vague about what we mean by words like "approach" and "close".

Isaac Newton (1642-1727) did not have a rigorous concept of limit. He called them "fluxions" and it made sense to him. It was not until around 1820 French Mathematician Augustin-Louis Cauchy (1789 – 1857), some 150 years after Gottfried Leibniz and Isaac Newton developed calculus, that the modern idea of limit was invented. This formal definition of limit led to great advances in the development of calculus and eliminated gross mathematical errors even made by the best mathematicians in history like Swiss mathematician Leonard Euler.

Here is the formal definition of *limit*:

We define $\lim_{x \to c} f(x)$ to be the number L (if one exists) such that for every positive number ε (epsilon) > 0 (as small as we want), there is a positive number δ (delta) > 0 (sufficiently small) such that if $|x-c| < \delta$ and $x \neq c$ then $|f(x) - L| < \varepsilon$.

The following figure will help us with what this definition means:

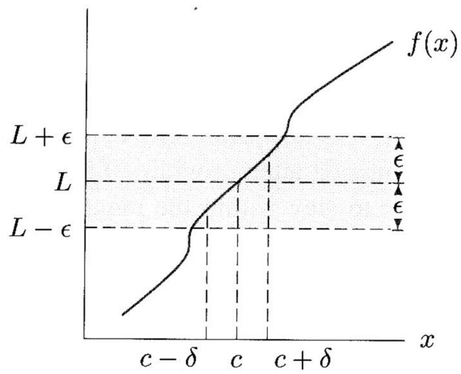


Figure 1.78: What the definition of the limit means graphically

When we say "*f(x)*" is close to *L*" we measure closeness by the distance between *f(x)* and *L.* $|f(x)-L|$ = Distance between *f(x)* and *L.*

When we say "as close to *L* as we want," we use the ε (the Greek letter epsilon) to specify how close.

We write $|f(x)-L|<\varepsilon$ to indicate that we want the distance between *f(x)* and *L* to be less than ε.

Similarly, we interpret "*x* is sufficiently close to *c*" as specifying a distance between *x* and *c*: $|x-c|<\delta$, where δ (the Greek letter delta) tells us how close *x* should be to *c.*

If $\lim_{x\to c} f(x)=L$, then we know that no matter how narrow the horizontal band determined by ε, there is always a δ which makes the graph stay within that band for $c-\delta<x<c+\delta$.

Basically what we are trying to do is can we guarantee that the inputs (sufficiently close to the value we are approaching but not equal to the value) will make the outputs as close to L as we want.

We will use a graphic illustration to help make sense of this so let's go back to $f(x)=\dfrac{\sin\theta}{\theta}$.

How close should θ be to 0 $(\delta = ? > 0)$ in order to make $\frac{\sin\theta}{\theta}$ within 0.01 of 1? $(\varepsilon = 0.01 > 0)$

First, set the y-range to go from y min = 0.99 to y max = 1.01. (0.99 < y < 1.01)

to get $|2x-6| < \varepsilon$ would require that $2|x-3| < \varepsilon$ or $|x-3| < \frac{\varepsilon}{2}$.

Since $|x-c| < d$ then $\delta = \frac{\varepsilon}{2}$.

One- and Two-Sided Limits

When we write $\lim_{x\to 2} f(x)$ we mean that the number *f(x)* approaches as *x* approaches 2 from *both sides.* This is a *Two-Sided Limit*.

If we want *x* to approach 2 only through values greater than 2 (like 2.1, 2.01, 2.003), we write $\lim_{x\to 2^+} f(x)$. This is called a *right-hand limit*. (Similar to the concept of right difference quotient)

If we want *x* to approach 2 only through values less than 2 (like 1.9, 1.99, 1.994), we write $\lim_{x\to 2^-} f(x)$. This is called a *left-hand limit*. (Similar to the concept of left difference quotient)

Right-hand limits and *left-hand limits* are examples of *One-Sided Limits*.

If both the left-hand and right-hand limits are equal, then it can be proved that $\lim_{x\to 2} f(x)$ exists.

Whenever there is no number L that $\lim_{x \to c} f(x) = L$, we say $\lim_{x \to c} f(x)$ does not exist.

***Limits have to be a number and it has to be unique for that function**

Examples of Limits That Do Not Exist

1) **Right – Hand Limit and Left-Hand Limit are different.**
   The one-sided limits exist but are different. At any integer, for example, the greatest integer function doesn't have a limit. Functions with split definitions can fall in this category at the point where the split occurs.

   For example, with $\lim_{x \to 2}\left(\frac{|x-2|}{x-2}\right)$,

   $f(x) = \begin{cases} x-1 \text{ for } x<1 \\ x^2 \text{ for } x \geq 1 \end{cases}$, $\lim_{x \to 1} f(x)$ doesn't exist.

2) **The function does not approach any finite number L as $x \to c$.**
   The outputs grow without bound as the inputs approaches the point from either one side or the other, or both. For example.

The Derivative

The concept of **Derivative** is at the core of Calculus and modern mathematics. The definition of the derivative can be approached in two different ways. One is geometrical (as a slope of a curve) and the other one is physical (as a rate of change). Historically there was (and maybe still is) a fight between mathematicians which of the two illustrates the concept of the derivative best and which one is more useful. We will not dwell on this and will introduce both concepts. Our emphasis will be on the use of the derivative as a tool.

The Physical Concept of the Derivative

This approach was used by Newton in the development of his Classical Mechanics. The main idea is the concept of velocity and speed. Indeed, assume you are traveling from point A to point B, what is the average velocity during the trip? It is given by

$$\text{Average velocity} = \frac{\text{distance from A to B}}{\text{time to get from A to B}}$$

If we now assume that A and B are very close to each other, we get close to what is called the **instantaneous velocity**. Of course, if A and B are close to each other, then the time it takes to travel from A to B will also be small. Indeed, assume that at time *t*=*a*, we are at A. If the time elapsed to get to

B is Δt, then we will be at B at time $t = a + \Delta t$. If Δs is the distance from A to B, then the average velocity is

$$\text{Average velocity} = \frac{\Delta s}{\Delta t}.$$

The instantaneous velocity (at A) will be found when Δt get smaller and smaller. Here we naturally run into the concept of limit. Indeed, we have

$$\text{Instantaneous Velocity (at A)} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

If *f*(*t*) describes the position at time *t*, then $\Delta s = f(a + \Delta t) - f(a)$. In this case, we have

$$\text{Instantaneous Velocity (at A)} = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

Example. Consider a parabolic motion given by the function $f(t) = t^2$. The instantaneous velocity at $t=a$ is given by

$$\lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = \lim_{\Delta t \to 0} \frac{(a + \Delta t)^2 - a^2}{\Delta t}.$$

Since

$$\frac{(a + \Delta t)^2 - a^2}{\Delta t} = \frac{2a\Delta t + \Delta t^2}{\Delta t} = 2a + \Delta t,$$

we conclude that the instantaneous velocity at $t=a$ is 2 a.

This concept of velocity may be extended to find the rate of change of any variable with respect to any other variable. For example, the volume of a gas depends on the temperature of the gas. So in this case, the variables are V (for volume) as a function of T (the temperature). In general, if we have $y = f(x)$, then the average rate of change of y with respect to x from $x = a$ to $x = a + \Delta x$, where $\Delta x \neq 0$, is

$$\text{Average Rate} = \frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

As before, the instantaneous rate of change of *y* with respect to *x* at $x = a$, is

$$\text{Instantaneous Velocity (at } x = a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

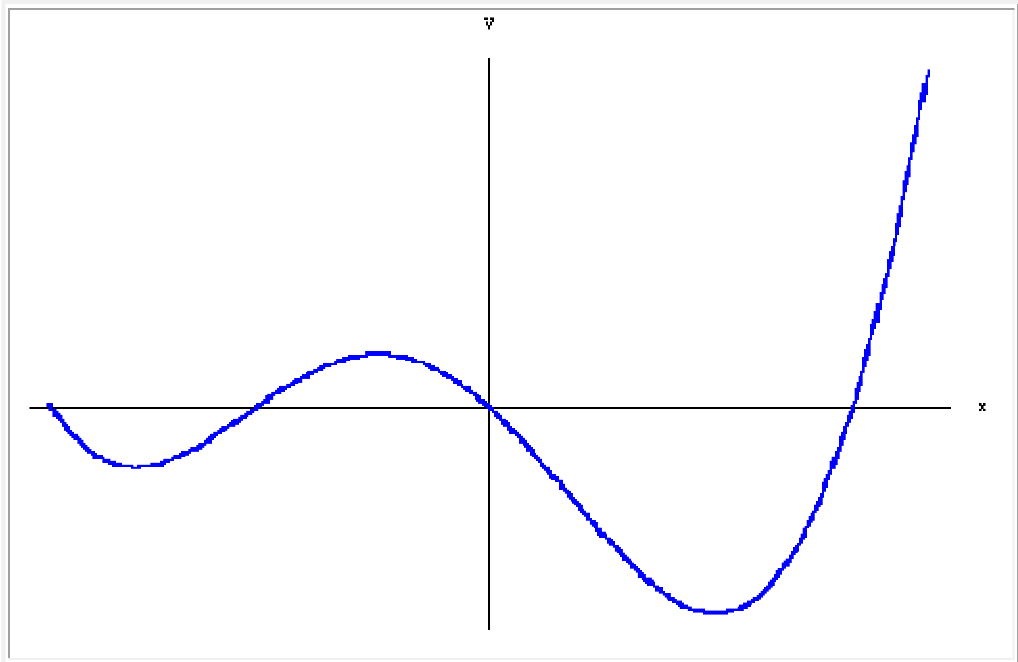
Notation. Now we get to the hardest part. Since we can not keep on writing "Instantaneous Velocity" while doing computations, we need to come up with a suitable notation for it. If we write *dx* for Δx small, then we can use the notation

$$\text{Instantaneous Velocity (at } x = a) = \frac{dy}{dx}(a).$$

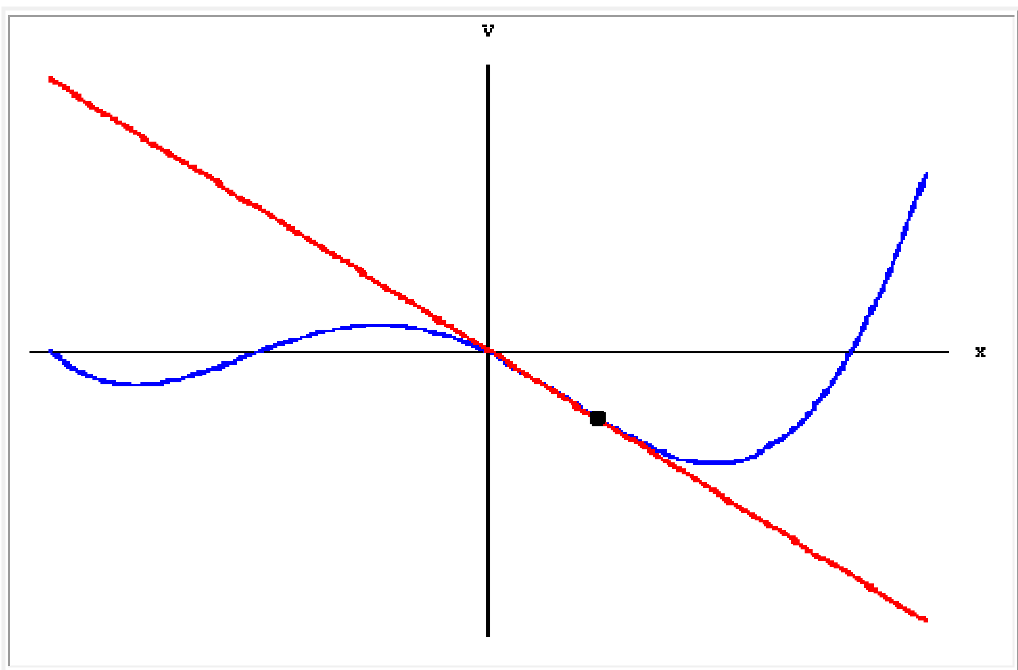
This is the notation introduced by Leibniz. (Wilhelm Gottfried Leibniz (1646-1716) and Isaac Newton (1642-1727) are considered the inventors of Calculus.)

The Geometrical Concept of the Derivative

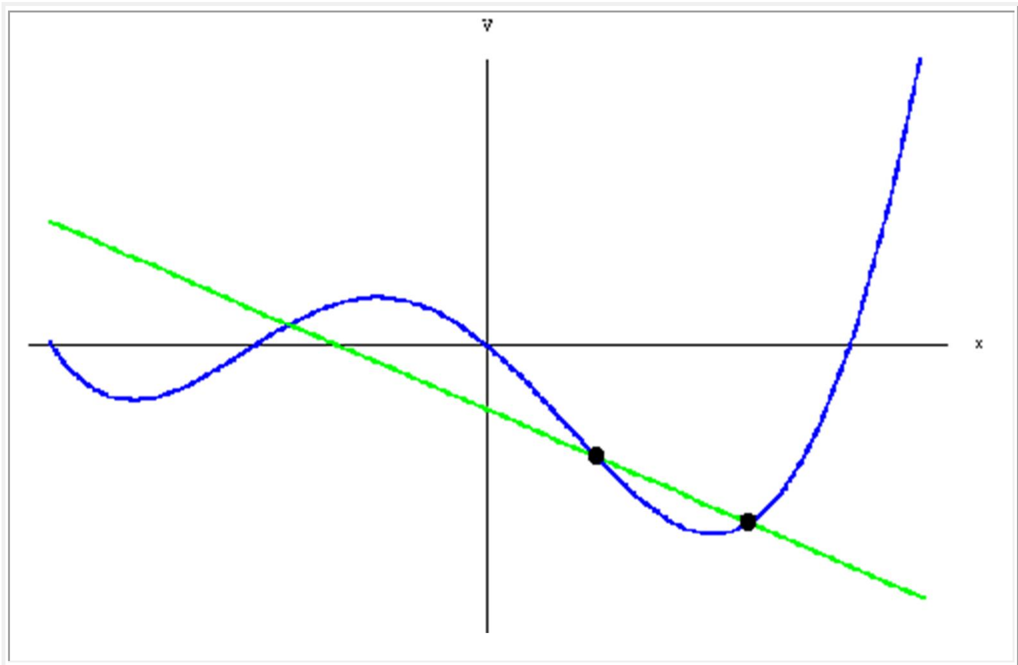
Consider a function $y = f(x)$ and its graph. Recall that the graph of a function is a set of points (that is $(x,f(x))$ for *x*'s from the domain of the function *f*). We may draw the graph in a plane with a horizontal axis (usually called the x-axis) and a vertical axis (usually called the y-axis).



Fix a point on the graph, say $(x_0, f(x_0))$. If the graph as a geometric figure is "nice" (i.e. smooth) around this point, it is natural to ask whether one can find the equation of the straight line "touching" the graph at that point. Such a straight line is called the **tangent line** at the point in question. The concept of tangent may be viewed in a more general framework.



(Note that the tangent line may not exist. We will discuss this case later on.) One way to find the tangent line is to consider points $(x,f(x))$ on the graph, where x is very close to x_0. Then draw the straight-line joining both points (see the picture below):



As you can see, when x get closer and closer to x_0, the lines get closer and closer to the tangent line. Since all these lines pass through the point $(x_0,f(x_0))$, their equations will be determined by finding their slope: The slope of the line passing through the points $(x_0,f(x_0))$ and $(x,f(x))$ (where $x \neq x_0$) is given by

$$m(x) = \frac{f(x) - f(x_0)}{x - x_0}.$$

The tangent itself will have a slope m, which is very close to $m(x)$ when x itself is very close to x_0. This is the concept of limit once again!

In other words, we have

$$m = \lim_{x \to x_0} m(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

So the equation of the tangent line is

$$y - f(x_0) = m(x - x_0).$$

Notation. Writing "m" for the slope of the tangent line does not carry enough information; we want to keep track of the function *f*(*x*) and the point x_0 in our notation. The common notation used is

$$m = f'(x_0).$$

In this case, the equation of the tangent line becomes

$$y - f(x_0) = f'(x_0)\,(x - x_0)$$

where

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

One last remark: Sometimes it is more convenient to compute limits when the variable approaches 0. One way to do that is to make a translation along the x-axis. Indeed, if we set $h = x - x_0$, we get

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Derivative as a Rate Measure and Measure of slope;

Differentiation is a method to compute the rate at which a dependent output *y* changes with respect to the change in the independent input *x*. This rate of change is called the *derivative* of *y* with respect to *x*. In more precise language, the dependence of *y* upon *x* means that *y* is a **function** of *x*. This functional relationship is often denoted $y = f(x)$, where *f* denotes the function. If *x* and *y* are **real numbers**, and if the **graph** of *y* is plotted against *x*, the derivative measures the **slope** of this graph at each point.

The simplest case is when *y* is a **linear function** of *x*, meaning that the graph of *y* divided by *x* is a line. In this case, $y = f(x) = m\,x + b$, for real numbers *m* and *b*, and the slope *m* is given by

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x},$$

where the symbol φ (the uppercase form of the Greek letter **Delta**) is an abbreviation for "change in." This formula is true because

$$y + \Delta y = f(x + \Delta x) = m\,(x + \Delta x) + b = m\,x + m\,\Delta x + b = y + m\,\Delta x.$$

It follows that φ$y = m$ φx.

This gives an exact value for the slope of a line. If the function *f* is not linear (i.e. its graph is not a line), however, then the change in *y* divided by the change in *x* varies: differentiation is a method to find an exact value for this rate of change at any given value of *x*.

Rate of change as a limit value

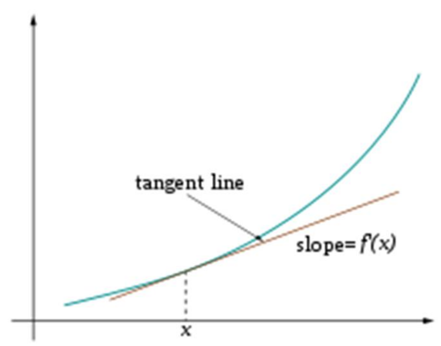


Figure 1. The **tangent** line at $(x, f(x))$

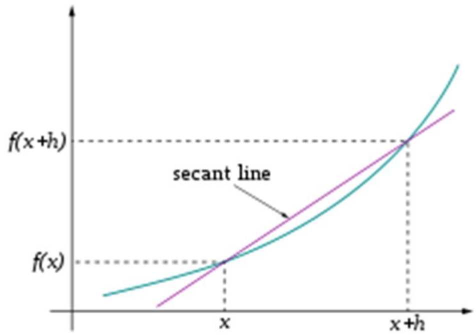


Figure 2. The **secant** to curve *y*= *f*(*x*) determined by points (*x*, *f*(*x*)) and (*x*+*h*, *f*(*x*+*h*))

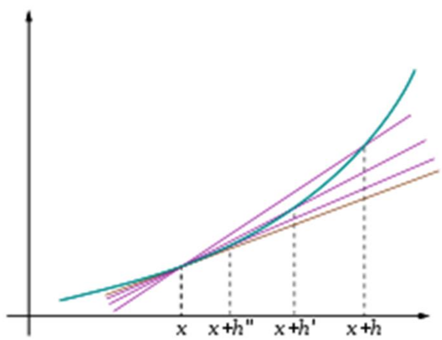


Figure 3. The tangent line as limit of secants

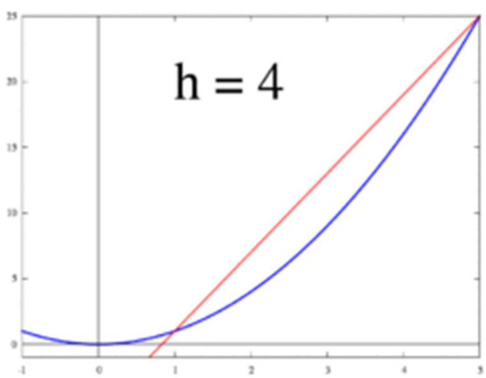


Figure 4. Animated illustration: the tangent line (derivative) as the limit of secants

The idea, illustrated by Figures 1 to 3, is to compute the rate of change as the **limit value** of the **ratio of the differences** Δy / Δx as Δx becomes infinitely small.

Notation

Calculus often employs two distinct notations for the same concept, one deriving from **Newton** and the other from **Leibniz**. In **Leibniz's notation**, an **infinitesimal** change in x is denoted by dx, and the derivative of y with respect to x is written

$$\frac{dy}{dx}$$

suggesting the ratio of two infinitesimal quantities. (The above expression is read as "the derivative of y with respect to x", "d y by d x", or "d y over d x". The oral form "d y d x" is often used conversationally, although it may lead to confusion.)

In **Lagrange's notation**, the instantaneous, limit value of the rate of change of a function $f(x)$ is designated $f'(x)$.

Rigorous definition

The most common approach to turn this intuitive idea into a precise definition is to define the derivative as a **limit** of difference quotients of real numbers. This is the approach described below.

Let f be a real valued function defined in an **open neighborhood** of a real number a. In classical geometry, the tangent line to the graph of the function f at a was the unique line through the point $(a, f(a))$ that did *not* meet the graph of f **transversally**, meaning that the line did not pass straight through the graph. The derivative of y with respect to x at a is, geometrically, the slope of the tangent line to the graph of f at a. The slope of the tangent line is very close to the slope of the line through $(a, f(a))$ and a nearby point on the graph, for example $(a + h, f(a + h))$. These lines are called **secant lines**. A value of h close to zero gives a good approximation to the slope of the tangent line, and smaller values (in **absolute value**) of h will, in general, give better **approximations**. The slope m of the secant line is the difference between the y values of these points divided by the difference between the x values, that is,

$$m = \frac{\Delta f(a)}{\Delta a} = \frac{f(a+h) - f(a)}{(a+h) - (a)} = \frac{f(a+h) - f(a)}{h}.$$

This expression is **Newton**'s **difference quotient**. The derivative is the value of the difference quotient as the secant lines approach the tangent line. Formally, the derivative of the function f at a is the **limit**

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

of the difference quotient as h approaches zero, if this limit exists. If the limit exists, then f is ***differentiable*** at a. Here $f'(a)$ is one of several common notations for the derivative (**see below**).

Equivalently, the derivative satisfies the property that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a) \cdot h}{h} = 0,$$

which has the intuitive interpretation (see Figure 1) that the tangent line to f at a gives the *best* linear *approximation*

$$f(a+h) \approx f(a) + f'(a)h$$

to f near a (i.e., for small h). This interpretation is the easiest to generalize to other settings **Substituting** 0 for h in the difference quotient causes **division by zero**, so the slope of the tangent line cannot be found directly using this method. Instead, define $Q(h)$ to be the difference quotient as a function of h:

$$Q(h) = \frac{f(a+h) - f(a)}{h}.$$

$Q(h)$ is the slope of the secant line between $(a, f(a))$ and $(a + h, f(a + h))$. If f is a **continuous function**, meaning that its graph is an unbroken curve with no gaps, then Q is a continuous function away from $h = 0$. If the limit $\lim_{h \to 0} Q(h)$ exists, meaning that there is a way of choosing a value for $Q(0)$ that makes Q a continuous function, then the function f is differentiable at a, and its derivative at a equals $Q(0)$.

In practice, the existence of a continuous extension of the difference quotient $Q(h)$ to $h = 0$ is shown by modifying the numerator to cancel h in the denominator. Such manipulations can make the limit value of Q for small h clear even though Q is still not defined at $h = 0$. This process can be long and tedious for complicated functions, and many shortcuts are commonly used to simplify the process.

Example

The squaring function $f(x) = x^2$ is differentiable at $x = 3$, and its derivative there is 6. This result is established by calculating the limit as h approaches zero of the difference quotient of $f(3)$:

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} = \lim_{h \to 0}$$

The last expression shows that the difference quotient equals $6 + h$ when $h \neq 0$ and is undefined when $h = 0$, because of the definition of the difference quotient. However, the definition of the limit says the difference quotient does not need to be defined when $h = 0$. The limit is the result of letting h go to zero, meaning it is the value that $6 + h$ tends to as h becomes very small:

$$\lim_{h\to 0}(6+h) = 6+0 = 6.$$

Hence the slope of the graph of the squaring function at the point (3, 9) is 6, and so its derivative at $x = 3$ is $f'(3) = 6$.

More generally, a similar computation shows that the derivative of the squaring function at $x = a$ is $f'(a) = 2a$.

The derivative as a function

Let f be a function that has a derivative at every point a in the **domain** of f. Because every point a has a derivative, there is a function that sends the point a to the derivative of f at a. This function is written $f'(x)$ and is called the *derivative function* or the *derivative* of f. The derivative of f collects all the derivatives of f at all the points in the domain of f.

Sometimes f has a derivative at most, but not all, points of its domain. The function whose value at a equals $f'(a)$ whenever $f'(a)$ is defined and elsewhere is undefined is also called the derivative of f. It is still a function, but its domain is strictly smaller than the domain of f.

Using this idea, differentiation becomes a function of functions: The derivative is an **operator** whose domain is the set of all functions that have derivatives at every point of their domain and whose range is a set of functions. If we denote this operator by D, then $D(f)$ is the function $f'(x)$. Since $D(f)$ is a function, it can be evaluated at a point a. By the definition of the derivative function, $D(f)(a) = f'(a)$.

For comparison, consider the doubling function $f(x) = 2x$; f is a real-valued function of a real number, meaning that it takes numbers as inputs and has numbers as outputs:

$$1 \mapsto 2,$$
$$2 \mapsto 4,$$
$$3 \mapsto 6.$$

The operator D, however, is not defined on individual numbers. It is only defined on functions:

$$D(x \mapsto 1) = (x \mapsto 0),$$
$$D(x \mapsto x) = (x \mapsto 1),$$
$$D(x \mapsto x^2) = (x \mapsto 2 \cdot x).$$

Because the output of D is a function, the output of D can be evaluated at a point. For instance, when D is applied to the squaring function,

$$x \mapsto x^2,$$

D outputs the doubling function,

$$x \mapsto 2x,$$

which we named $f(x)$. This output function can then be evaluated to get $f(1) = 2$, $f(2) = 4$, and so on

Functions of more than one variable; Partial Derivatives:-

In **mathematics**, a **partial derivative** of a **function** of several variables is its **derivative** with respect to one of those variables, **with the others held constant** (as opposed to the **total derivative**, in which all variables are allowed to vary). Partial derivatives are used in **vector calculus** and **differential geometry**.

The partial derivative of a function f with respect to the variable x is variously denoted by

$$f'_x,\ f_x,\ \partial_x f,\ \frac{\partial}{\partial x}f,\ \text{or}\ \frac{\partial f}{\partial x}$$

The partial-derivative symbol is Ö One of the first known uses of the symbol in mathematics is by **Marquis de Condorcet** from 1770, who used it for partial differences. The modern **partial derivative** notation is by **Adrien-Marie Legendre** (1786), though he later abandoned it; **Carl Gustav Jacob Jacobi** re-introduced the symbol

Basic definition

The function f can be reinterpreted as a family of functions of one variable indexed by the other variables:

$$f(x,y) = f_x(y) = x^2 + xy + y^2.$$

In other words, every value of x defines a function, denoted f_x, which is a function of one variable That is,

$$f_x(y) = x^2 + xy + y^2.$$

Once a value of x is chosen, say a, then $f(x,y)$ determines a function f_a which sends y to $a^2 + ay + y^2$:

$$f_a(y) = a^2 + ay + y^2.$$

In this expression, a is a *constant*, not a *variable*, so f_a is a function of only one real variable, that being y. Consequently, the definition of the derivative for a function of one variable applies:

$$f_a'(y) = a + 2y.$$

The above procedure can be performed for any choice of a. Assembling the derivatives together into a function gives a function which describes the variation of f in the y direction:

$$\frac{\partial f}{\partial y}(x,y) = x + 2y.$$

This is the partial derivative of f with respect to y. Here Öis a rounded d called the **partial derivative symbol**. To distinguish it from the letter d, Öis sometimes pronounced "del" or "partial" instead of "dee".

In general, the **partial derivative** of a function $f(x_1,...,x_n)$ in the direction x_i at the point $(a_1,...,a_n)$ is defined to be:

$$\frac{\partial f}{\partial x_i}(a_1,\ldots,a_n) = \lim_{h\to 0}\frac{f(a_1,\ldots,a_i+h,\ldots,a_n) - f(a_1,\ldots,a_i,\ldots,a_n)}{h}.$$

In the above difference quotient, all the variables except x_i are held fixed. That choice of fixed values determines a function of one variable $f_{a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n}(x_i) = f(a_1,\ldots,a_{i-1},x_i,a_{i+1},\ldots,a_n)$, and by definition,

$$\frac{df_{a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n}}{dx_i}(a_i) = \frac{\partial f}{\partial x_i}(a_1,\ldots,a_n).$$

In other words, the different choices of a index a family of one-variable functions just as in the example above. This expression also shows that the computation of partial derivatives reduces to the computation of one-variable derivatives.

An important example of a function of several variables is the case of a **scalar-valued function** $f(x_1,...x_n)$ on a domain in Euclidean space $\mathbf{R}^n$ (e.g., on $\mathbf{R}^2$ or $\mathbf{R}^3$). In this case f has a partial derivative $\partial f/\partial x_j$ with respect to each variable x_j. At the point a, these partial derivatives define the vector

$$\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a)\right).$$

This vector is called the gradient of f at a. If f is differentiable at every point in some domain, then the gradient is a vector-valued function ∇f which takes the point a to the vector $\nabla f(a)$. Consequently, the gradient produces a **vector field**.

A common **abuse of notation** is to define the **del operator** (∇) as follows in three-dimensional **Euclidean space** $\mathbf{R}^3$ with **unit vectors** $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$:

$$\nabla = \left[\frac{\partial}{\partial x}\right]\hat{\mathbf{i}} + \left[\frac{\partial}{\partial y}\right]\hat{\mathbf{j}} + \left[\frac{\partial}{\partial z}\right]\hat{\mathbf{k}}$$

Or, more generally, for n-dimensional Euclidean space $\mathbf{R}^n$ with coordinates $(x_1, x_2, x_3,...,x_n)$ and unit vectors ($\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \ldots, \hat{\mathbf{e}}_n$):

$$\nabla = \sum_{j=1}^{n}\left[\frac{\partial}{\partial x_j}\right]\hat{\mathbf{e}}_j = \left[\frac{\partial}{\partial x_1}\right]\hat{\mathbf{e}}_1 + \left[\frac{\partial}{\partial x_2}\right]\hat{\mathbf{e}}_2 + \left[\frac{\partial}{\partial x_3}\right]\hat{\mathbf{e}}_3 + \cdots + \left[\frac{\partial}{\partial x_n}\right]\hat{\mathbf{e}}_n$$

Homogenous Functions and Euler's Theorem

Definition

Multivariate functions that are "homogeneous" of some degree are often used in economic theory. A function is homogeneous of degree k if, when each of its arguments is multiplied by any number $t > 0$, the value of the function is multiplied by t^k. For example, a function is homogeneous of degree 1 if, when all its arguments are multiplied by any number $t > 0$, the value of the function is multiplied by the same number t.

Here is a precise definition. Because the definition involves the relation between the value of the function at $(x_1, ..., x_n)$ and it value at points of the form $(tx_1, ..., tx_n)$ where t is any positive number, it is restricted to functions for which $(tx_1, ..., tx_n)$ is in the domain whenever $t > 0$ and $(x_1, ..., x_n)$ is in the domain. (Some domains that have this property are: the set of all real numbers, the set of nonnegative real numbers, the set of positive real numbers, the set of all n-tuples $(x_1, ..., x_n)$ of real numbers, the set of n-tuples of nonnegative real numbers, and the set of n-tuples of positive real numbers.)

Definition

A function f of n variables for which $(tx_1, ..., tx_n)$ is in the domain whenever $t > 0$ and $(x_1, ..., x_n)$ is in the domain is **homogeneous of degree** k if

$f(tx_1, ..., tx_n) = t^k f(x_1, ..., x_n)$ for all $(x_1, ..., x_n)$ in the domain of f and all $t > 0$.

Example

For the function $f(x_1, x_2) = Ax_1^a x_2^b$ with domain $\{(x_1, x_2): x_1 \geq 0 \text{ and } x_2 \geq 0\}$ we have

$f(tx_1, tx_2) = A(tx_1)^a(tx_2)^b = At^{a+b}x_1^a x_2^b = t^{a+b} f(x_1, x_2)$,

so that f is homogeneous of degree $a + b$.

Example

Let $f(x_1, x_2) = x_1 + x_2^2$, with domain $\{(x_1, x_2): x_1 \geq 0 \text{ and } x_2 \geq 0\}$. Then

$f(tx_1, tx_2) = tx_1 + t^2x_2^2$.

It doesn't *seem* to be possible to write this expression in the form $t^k(x_1 + x_2^2)$ for any value of k. But how do we *prove* that there is no such value of k? Suppose that there were such a value. That is, suppose that for some k we have

$tx_1 + t^2x_2^2 = t^k(x_1 + x_2^2)$ for all $(x_1, x_2) \geq (0, 0)$ and all $t > 0$.

Then in particular, taking $t = 2$, we have

$2x_1 + 4x_2 = 2^k(x_1 + x_2^2)$ for all (x_1, x_2).

Taking $(x_1, x_2) = (1, 0)$ and $(x_1, x_2) = (0, 1)$ we thus have

$2 = 2^k$ and $4 = 2^k$,

which is not possible. Thus f is not homogeneous of any degree.

In economic theory we often assume that a firm's production function is homogeneous of degree 1 (if all inputs are multiplied by t then output is multiplied by t). A production function with this property is said to have "constant returns to scale".

Suppose that a consumer's demand for goods, as a function of prices and her income, arises from her choosing, among all the bundles she can afford, the one that is best according to her preferences. Then we can show that this demand function is homogeneous of degree zero: if all prices and the consumer's income are multiplied by any number $t > 0$ then her demands for goods stay the same.

Partial derivatives of homogeneous functions

The following result is sometimes useful.

Proposition

Let f be a differentiable function of n variables that is homogeneous of degree k. Then each of its partial derivatives f'_i (for $i = 1, ..., n$) is homogeneous of degree $k - 1$.

Proof

The homogeneity of f means that

$f(tx_1, ..., tx_n) = t^k f(x_1, ..., x_n)$ for all $(x_1, ..., x_n)$ and all $t > 0$.

Now differentiate both sides of this equation with respect to x_i, to get

$t f'_i(tx_1, ..., tx_n) = t^k f'_i(x_1, ..., x_n)$,

and then divide both sides by t to get

$f'_i(tx_1, ..., tx_n) = t^{k-1} f'_i(x_1, ..., x_n)$,

so that f'_i is homogeneous of degree $k - 1$.

Application: level curves of homogeneous functions

This result can be used to demonstrate a nice result about the slopes of the **level curves** of a homogeneous function. As **we have seen**, the slope of the level curve of the function F through the point (x_0, y_0) at this point is

$$-\frac{F_1'(x_0, y_0)}{F_2'(x_0, y_0)}.$$

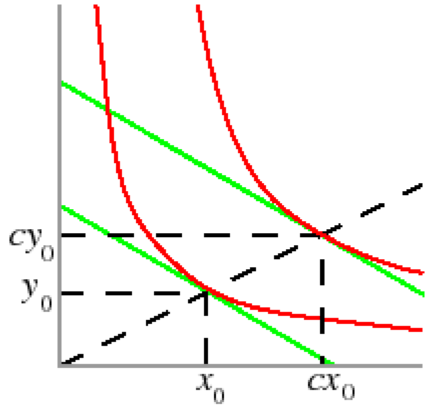
Now suppose that F is homogeneous of degree k, and consider the level curve through (cx_0, cy_0) for some number $c > 0$. At (cx_0, cy_0), the slope of this curve is

$$-\frac{F_1'(cx_0, cy_0)}{F_2'(cx_0, cy_0)}.$$

By the previous result, F'_1 and F'_2 are homogeneous of degree $k-1$, so this slope is equal to

$$-\frac{c^{k-1}F_1'(x_0, y_0)}{c^{k-1}F_2'(x_0, y_0)} = -\frac{F_1'(x_0, y_0)}{F_2'(x_0, y_0)}.$$

That is, the slope of the level curve through (cx_0, cy_0) at the point (cx_0, cy_0) is exactly the same as the slope of the level curve through (x_0, y_0) at the point (x_0, y_0), as illustrated in the following figure.



In this figure, the red lines are two level curves, and the two green lines, the tangents to the curves at (x_0, y_0) and at (cx_0, xy_0), are parallel.

We may summarize this result as follows.

Let F be a differentiable function of two variables that is homogeneous of some degree. Then along any given ray from the origin, the slopes of the level curves of F are the same.

Euler's theorem

A function homogeneous of some degree has a property sometimes used in economic theory that was first discovered by **Leonhard Euler (1707-1783)**.

Proposition (Euler's theorem)

The differentiable function f of n variables is homogeneous of degree k if and only if $\sum_{i=1}^{n} x_i f_i'(x_1, ..., x_n) = k f(x_1, ..., x_n)$ for all $(x_1, ..., x_n)$. (*)

Condition (*) may be written more compactly, using the notation ∇f for the **gradient vector** of f and letting $x = (x_1, ..., x_n)$, as

$x \cdot \nabla f(x) = k f(x)$ for all x.

Proof

I first show that if f is homogeneous of degree k then (*) holds. If f is homogeneous of degree k then

$f(tx_1, ..., tx_n) = t^k f(x_1, ..., x_n)$ for all $(x_1, ..., x_n)$ and all $t > 0$.

Differentiate each side of this equation with respect to t, to give

$x_1 f'_1(tx_1, ..., tx_n) + x_2 f'_2(tx_1, ..., tx_n) + ... + x_n f'_n(tx_1, ..., tx_n) = kt^{k-1} f(x_1, ..., x_n)$.

Now set $t = 1$, to obtain (*).

I now show that if (*) holds then f is homogeneous of degree k. Suppose that (*) holds.

Fix $(x_1, ..., x_n)$ and define the function g of a single variable by

$g(t) = t^{-k} f(tx_1, ..., tx_n) - f(x_1, ..., x_n)$.

We have

$g'(t) = -kt^{-k-1} f(tx_1, ..., tx_n) + t^{-k}\sum_{i=1}^{n} x_i f'_i(tx_1, ..., tx_n)$.

By (*), we have

$\sum_{i=1}^{n} tx_i f_i'(tx_1, ..., tx_n) = k f(tx_1, ..., tx_n)$,

so that $g'(t) = 0$ for all t. Thus $g(t)$ is a constant. But $g(1) = 0$, so $g(t) = 0$ for all t, and hence $f(tx_1, ..., tx_n) = t^k f(x_1, ..., x_n)$ for all $t > 0$, so that f is homogeneous of degree k.

Example

Let $f(x_1, ..., x_n)$ be a firm's production function; suppose it is homogeneous of degree 1 (i.e. has "constant returns to scale"). Euler's theorem shows that if the price (in terms of units of output) of each input i is its "marginal product" $f'_i(x_1, ..., x_n)$, then the total cost, namely

$\sum_{i=1}^{n} x_i f_i'(x_1, ..., x_n)$

is equal to the total output, namely $f(x_1, ..., x_n)$.

Differentiation of Implicit functions:-

Inverse functions

A common type of implicit function is an **inverse function**. If f is a function, then the inverse function of f, called f^{-1}, is the function giving a solution of the equation

$x = f(y)$

for y in terms of x. This solution is

$$y = f^{-1}(x).$$

Intuitively, an inverse function is obtained from f by interchanging the roles of the dependent and independent variables. Stated another way, the inverse function gives the solution for y of the equation

$$R(x, y) = x - f(y) = 0.$$

Examples.

1. The **natural logarithm** ln(x) gives the solution y = ln(x) of the equation $x - e^y = 0$ or equivalently of $x = e^y$. Here $f(y) = e^y$ and $f^{-1}(x) = \ln(x)$.
2. The **product log** is an implicit function giving the solution for y of the equation $x - y\,e^y = 0$.

Algebraic functions

An **algebraic function** is a function that satisfies a polynomial equation whose coefficients are themselves polynomials. For example, an algebraic function in one variable x gives a solution for y of an equation

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_0(x) = 0$$

where the coefficients $a_i(x)$ are polynomial functions of x. Algebraic functions play an important role in **mathematical analysis** and **algebraic geometry**. A simple example of an algebraic function is given by the unit circle equation:

$$x^2 + y^2 - 1 = 0.$$

Solving for y gives an explicit solution:

$$y = \pm\sqrt{1 - x^2}.$$

But even without specifying this explicit solution, it is possible to refer to the implicit solution of the unit circle equation.

While explicit solutions can be found for equations that are quadratic, cubic, and quartic in y, the same is not in general true for quintic and higher degree equations, such as

$$y^5 + 2y^4 - 7y^3 + 3y^2 - 6y - x = 0.$$

Nevertheless, one can still refer to the implicit solution $y = g(x)$ involving the multi-valued implicit function g.

Implicit differentiation

In **calculus**, a method called **implicit differentiation** makes use of the **chain rule** to differentiate implicitly defined functions.

As explained in the introduction, y can be given as a function of x implicitly rather than explicitly. When we have an equation $R(x, y) = 0$, we may be able to solve it for y and then differentiate. However, sometimes it is simpler to differentiate $R(x, y)$ with respect to x and y and then solve for dy/dx.

Examples

1. Consider for example

$$y + x + 5 = 0$$

This function normally can be manipulated by using **algebra** to change this **equation** to one expressing *y* in terms of an **explicit function**:

$$y = -x - 5\,,$$

where the right side is the explicit function whose output value is *y*. Differentiation then gives *dy*/*dx* = ī 1. Alternatively, one can totally differentiate the original equation:

$$\frac{dy}{dx} + \frac{dx}{dx} + \frac{d}{dx}(5) = 0;$$

$$\frac{dy}{dx} + 1 = 0.$$

Solving for *dy*/*dx* gives:

$$\frac{dy}{dx} = -1,$$

the same answer as obtained previously.

2. An example of an implicit function, for which implicit differentiation might be easier than attempting to use explicit differentiation, is

$$x^4 + 2y^2 = 8$$

In order to differentiate this explicitly with respect to *x*, one would have to obtain (via algebra)

$$y = f(x) = \pm\sqrt{\frac{8 - x^4}{2}},$$

and then differentiate this function. This creates two derivatives: one for $y > 0$ and another for $y < 0$.

One might find it substantially easier to implicitly differentiate the original function:

$$4x^3 + 4y\frac{dy}{dx} = 0,$$

giving,

$$\frac{dy}{dx} = \frac{-4x^3}{4y} = \frac{-x^3}{y}$$

3. Sometimes standard explicit differentiation cannot be used and, in order to obtain the derivative, implicit differentiation must be employed. An example of such a case is the equation y^5 ī y = x. It is impossible to express y explicitly as a function of x and therefore dy/dx cannot be found by explicit differentiation. Using the implicit method, dy/dx can be expressed:

$$5y^4\frac{dy}{dx} - \frac{dy}{dx} = \frac{dx}{dx}$$

where $dx/dx = 1$. Factoring out dy/dx shows that

$$\frac{dy}{dx}(5y^4 - 1) = 1$$

which yields the final answer

$$\frac{dy}{dx} = \frac{1}{5y^4 - 1},$$

which is defined for $y \neq \pm\frac{1}{\sqrt[4]{5}}.$

Formula for two variables

"The Implicit Function Theorem states that if F is defined on an open disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b)$ Í 0, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by"

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y},$$

where F_x and F_y indicate the derivatives of F with respect to x and y.

The above formula comes from using the **generalized chain rule** to obtain the **total derivative**ð with respect to *x*ð of both sides of $F(x, y) = 0$:

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0,$$

and hence

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

Implicit function theorem

It can be shown that if $R(x, y)$ is given by a smooth **sub manifold** M in $\mathbf{R}^2$, and (a, b) is a point of this submanifold such that the **tangent space** there is not vertical

(that is, $\frac{\partial R}{\partial y} \neq 0$), then M in some small enough **neighbourhood** of (a, b) is given by a **parametrization** $(x, f(x))$ where f is a **smooth function**. In less technical language, implicit functions exist and can be differentiated, unless the tangent to the supposed graph would be vertical. In the standard case where we are given an equation

$$R(x, y) = 0$$

the condition on R can be checked by means of **partial derivatives**.

marginal rate of substitution of the two goods: how much more of y one must receive in order to be indifferent to a loss of 1 unit of x.

A real-valued **function** f defined on a **domain** X has a **global** (or **absolute**) **maximum point** at x^* if $f(x^*)$ Ó$f(x)$ for all x in X. Similarly, the function has a **global** (or **absolute**) **minimum point** at x^* if $f(x^*)$ Ò$f(x)$ for all x in X. The value of the function at a maximum point is called the **maximum value** of the function and the value of the function at a minimum point is called the **minimum value** of the function.

If the domain X is a metric space then f is said to have a **local** (or **relative**) **maximum point** at the point x^* if there exists some $\varepsilon > 0$ such that $f(x^*)$ Ó$f(x)$ for all x in X within distance ε of x^*. Similarly, the function has a **local minimum point** at x^* if $f(x^*)$ Ò$f(x)$ for all x in X within distance ε of x^*. A similar definition can be used when X is a **topological space**, since the definition just given can be rephrased in terms of neighbourhoods. Note that a global maximum point is always a local maximum point, and similarly for minimum points.

In both the global and local cases, the concept of a **strict** extremum can be defined. For example, x^* is a **strict global maximum point** if, for all x in X with x Í x^*, we have $f(x^*) > f(x)$, and x^* is a **strict local maximum point** if there exists some $\varepsilon > 0$ such that, for all x in X within distance ε of x^* with x Í x^*, we have $f(x^*) > f(x)$. Note that a point is a strict global maximum point if and only if it is the unique global maximum point, and similarly for minimum points.

A **continuous** real-valued function with a **compact** domain always has a maximum point and a minimum point. An important example is a function whose domain is a closed (and bounded) **interval** of **real numbers** (see the graph above).

Finding functional maxima and minima

Finding global maxima and minima is the goal of **mathematical optimization**. If a function is continuous on a closed interval, then by the **extreme value theorem** global maxima and minima exist. Furthermore, a global maximum (or minimum) either must be a local maximum (or minimum) in the interior of the domain, or must lie on the boundary of the domain. So a method of finding a global maximum (or minimum) is to look at all the local maxima (or minima) in the interior, and also look at the maxima (or minima) of the points on the boundary; and take the biggest (or smallest) one.

Local extrema can be found by **Fermat's theorem**, which states that they must occur at **critical points**. One can distinguish whether a critical point is a local maximum or local minimum by using the **first derivative test**, **second derivative test**, or **higher-order derivative test**, given sufficient differentiability.

For any function that is defined **piecewise**, one finds a maximum (or minimum) by finding the maximum (or minimum) of each piece separately; and then seeing which one is biggest (or smallest).

Applications

Marginal rate of substitution

, when the level set $R(x, y) = 0$ is an **indifference curve** for the quantities x and y consumed of two goods, the absolute value of the implicit derivative is interpreted as the

UNIT-IV

Integration

What is integration?

The dictionary definition of *integration* is combining parts so that they work together or form a whole. Mathematically, integration stands for finding the area under a curve from one point to another. It is represented by

$$\int_a^b f(x)dx$$

where the symbol $\int$ is an integral sign, and a and b are the lower and upper limits of integration, respectively, the function f is the integrand of the integral, and x is the variable of integration. Figure 1 represents a graphical demonstration of the concept.

Riemann Sum

Let f be defined on the closed interval $[a,b]$, and let Δ be an arbitrary partition of $[a,b]$ such as: $a = x_0 < x_1 < x_2 < ..... < x_{n-1} < x_n = b$, where Δx_i is the length of the i^{th} subinterval (Figure 2).

If c_i is any point in the i^{th} subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, x_{i-1} \le c_i \le x_i$$

is called a Riemann sum of the function f for the partition Δ on the interval $[a,b]$. For a given partition Δ, the length of the longest subinterval is called the norm of the partition. It is denoted by $\|\Delta\|$ (the norm of Δ). The following limit is used to define the definite integral.

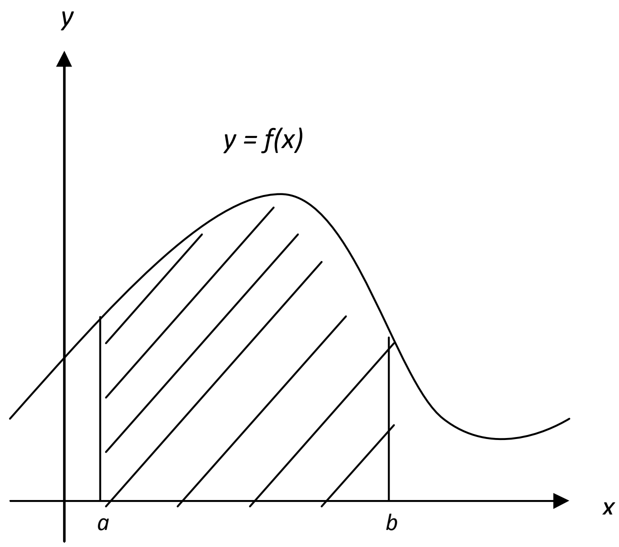


Figure The definite integral as the area of a region under the curve, $Area = \int_a^b f(x)dx$.

If c_i is any point in the i^{th} subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i,\ x_{i-1} \le c_i \le x_i$$

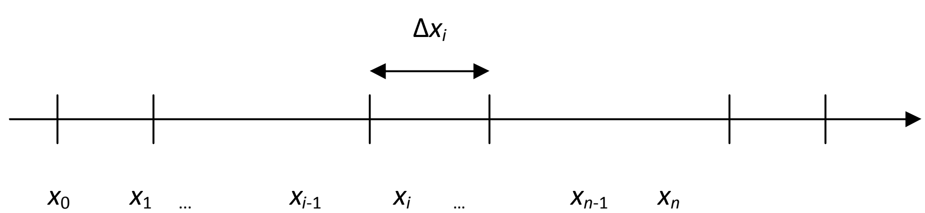


Figure Division of interval into n segments.

is called a Riemann sum of the function f for the partition Δ on the interval $[a,b]$. For a given partition Δ, the length of the longest subinterval is called the norm of the partition. It is denoted by $\|\Delta\|$ (the norm of Δ). The following limit is used to define the definite integral.

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = I$$

This limit exists if and only if for any positive number ε, there exists a positive number δ such that for every partition Δ of $[a,b]$ with $\|\Delta\| < \delta$, it follows that

$$\left| I - \sum_{i=1}^{n} f(c_i)\Delta x_i \right| < \varepsilon$$

for any choice of c_i in the i^{th} subinterval of Δ.

If the limit of a Riemann sum of f exists, then the function f is said to be integrable over $[a,b]$ and the Riemann sum of f on $[a,b]$ approaches the number I.

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = I$$

where

$$I = \int_a^b f(x)dx$$

Example

Find the area of the region between the parabola $y = x^2$ and the x-axis on the interval $[0,4.5]$. Use Riemann's sum with four partitions.

Solution

We evaluate the integral for the area as a limit of Riemann sums. We sketch the region (Figure 3), and partition $[0,4.5]$ into four subintervals of length

$$\Delta x = \frac{4.5-0}{4} = 1.125 .$$

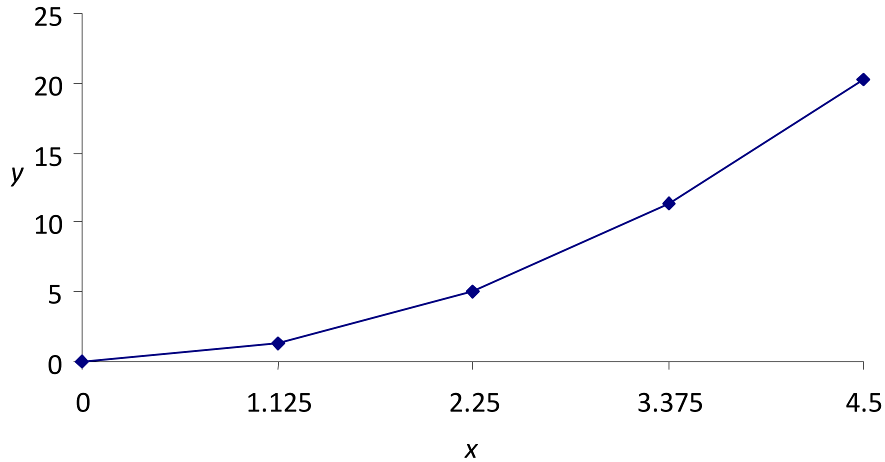


Figure Graph of the function $y = x^2$.

The points of partition are

$$x_0 = 0,\ x_1 = 1.125,\ x_2 = 2.25,\ x_3 = 3.375,\ x_4 = 4.5$$

Let's choose c_i's to be right hand endpoint of its subinterval. Thus,

$$c_1 = x_1 = 1.125,\ c_2 = x_2 = 2.25,\ c_3 = x_3 = 3.375,\ c_4 = x_4 = 4.5$$

The rectangles defined by these choices have the following areas:

$$f(c_1)\Delta x = f(1.125)\times(1.125) = (1.125)^2(1.125) = 1.4238$$

$$f(c_2)\Delta x = f(2.25)\times(1.125) = (2.25)^2(1.125) = 5.6953$$

$$f(c_3)\Delta x = f(3.375)\times(1.125) = (3.375)^2(1.125) = 12.814$$

$$f(c_4)\Delta x = f(4.5)\times(1.125) = (4.5)^2(1.125) = 22.781$$

The sum of the areas then is

$$\int_0^{4.5} x^2 dx \approx \sum_{i=1}^{4} f(c_i)\Delta x,$$

$$= 1.4238 + 5.6953 + 12.814 + 22.781$$

$$= 42.715$$

How does this compare with the exact value of the integral $\int_0^{4.5} x^2 dx$?

Example

Find the exact area of the region between the parabola $y = x^2$ and the $x-\text{axis}$ on the interval $[0, b]$. Use Riemann's sum.

Solution

Note that in Example 1 for $y = x^2$ that

$$f(c_i)\Delta x = i^2(\Delta x)^3$$

Thus, the sum of these areas, if the interval is divided into n equal segments is

$$S_n = \sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \sum_{i=1}^{n} i^2 (\Delta x)^3$$

$$= (\Delta x)^3 \sum_{i=1}^{n} i^2$$

Since

$\Delta x = \dfrac{b}{n}$, and

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

then

$$S_n = \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{b^3}{6}\frac{2n^2+n+2n+1}{n^2}$$

$$= \frac{b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$$

The definition of a definite integral can now be used

$$\int_a^b f(x)dx = \lim_{\|\Delta x\|\to 0}\sum_{i=1}^{n} f(c_i)\Delta x$$

To find the area under the parabola from $x=0$ to $x=b$, we have

$$\int_0^b x^2 dx = \lim_{|\Delta|\to 0}\sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty}\frac{b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$$

$$= \frac{b^3}{6}(2+0+0)$$

$$= \frac{b^3}{3}$$

For the value of $b=4.5$ as given in Example 1,

$$\int_0^{4.5} x^2 dx = \frac{4.5^3}{3}$$

= 30.375

The Mean Value Theorem for Integrals

The area of a region under a curve is usually greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle. The mean value theorem for integrals states that somewhere between these two rectangles, there exists a rectangle whose area is exactly equal to the area of the region under the curve, as shown in Figure 4. Another variation states that if a function f is continuous between a and b, then there is at least one point in $[a,b]$ where the function equals the average value of the function f over $[a,b]$.

Theorem: If the function f is continuous on the closed interval $[a,b]$, then there exists a number c in $[a,b]$ such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Example

Graph the function $f(x) = (x-1)^2$, and find its average value over the interval $[0,3]$. At what point in the given interval does the function assume its average value?

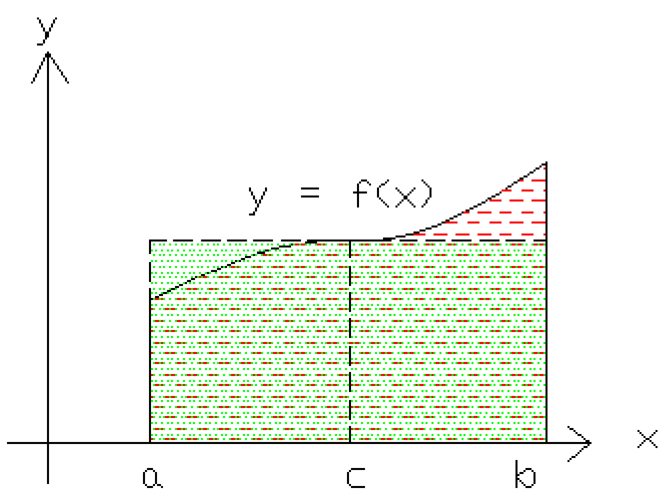


Figure Mean value rectangle.

Solution

$$Average(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

$$= \frac{1}{3-0}\int_0^3 (x-1)^2 dx$$

$$= \frac{1}{3}\int_0^3 (x^2 - 2x + 1) dx$$

$$= \frac{1}{3}\left[\left(\frac{1}{3}\times 27 - 9 + 3\right) - 0\right]$$

$$= 1$$

The average value of the function f over the interval $[0,3]$ is 1. Thus, the function assumes its average value at

$$f(c) = 1$$

$$(c-1)^2 = 1$$

$$c = 0,\ 2$$

The connection between integrals and area can be exploited in two ways. When a formula for the area of the region between the x-axis and the graph of a continuous function is known, it can be used to evaluate the integral of the function. However, if the area of region is not known, the integral of the function can be used to define and calculate the area. Table 1 lists a number of standard indefinite integral forms.

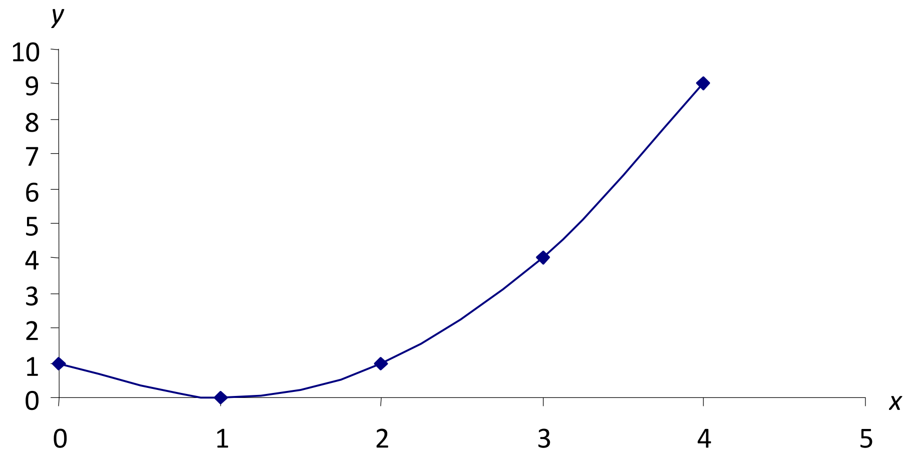


Figure The function $f(x) = (x-1)^2$.

Example 4

Find the area of the region between the circle $x^2 + y^2 = 1$ and the x-axis on the interval $[0,1]$ (the shaded region) in two different ways.

Solution

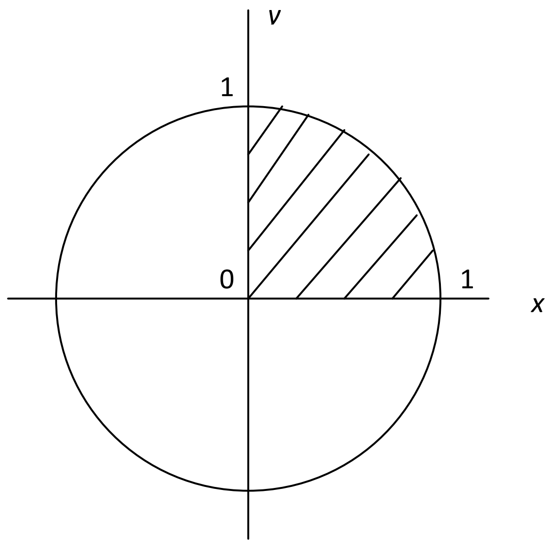


Figure Graph of the function $x^2 + y^2 = 1$.

The first and easy way to solve this problem is by recognizing that it is a quarter circle. Hence the area of the shaded area is

$$A = \frac{1}{4}\pi r^2$$

$$= \frac{1}{4}\pi(1)^2$$

$$= \frac{\pi}{4}$$

The second way is to use the integrals and the trigonometric functions. First, let's simplify the function $x^2 + y^2 = 1$.

$$x^2 + y^2 = 1$$
$$y^2 = 1 - x^2$$
$$y = \sqrt{1 - x^2}$$

The area of the shaded region is the equal to

$$A = \int_0^1 \sqrt{1-x^2}\,dx$$

We set $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$$A = \int_0^1 \sqrt{1-x^2}\,dx$$

$$= \int_0^{\pi/2} \sqrt{\left(1-\sin^2\theta\right)}\cos\theta\ d\theta$$

$$= \int_0^{\pi/2} \sqrt{\left(\cos^2\theta\right)}\cos\theta\ d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta\ d\theta$$

By using the following formula

$$\cos^2\theta = \frac{1+\cos 2\theta}{2},$$

we have

$$A = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right)d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= \left(\frac{\pi}{4} + 0\right) - \left(0+0\right)$$

$$= \frac{\pi}{4}$$

The following are some more examples of exact integration. You can use the brief table of integrals given in Table 1.

Table A brief table of integrals

$\int dx = x + C$	$\int \sin x dx = -\cos x + C$
$\int a\, f(x)\, dx = a \int f(x)\, dx + C$	$\int \cos x dx = \sin x + C$
$\int [u(x) \pm v(x)] dx = \int u(x) dx \pm \int v(x) dx + C$	$\int \tan x dx = -\ln\lvert\cos x\rvert + C = \ln\lvert\sec x\rvert + C$
$\int x^n dx = \frac{x^{n+1}}{n+1} + C$	$\int \sec(ax) dx = \frac{1}{a} \ln\lvert\sec(ax) + \tan(ax)\rvert + C$
$\int u\, dv = u\, v - \int v\, du + C$	$\int \cot x dx = -\ln\lvert\csc x\rvert + C = \ln\lvert\sin x\rvert + C$
$\int \frac{dx}{ax+b} = \frac{1}{a} \ln\lvert ax + b\rvert + C$	$\int \sec^2 ax dx = \frac{1}{a} \tan(ax) + C$
$\int a^x dx = \frac{a^x}{\ln a} + C$	$\int \sec(x) \tan(x) dx = \sec(x) + C$

$\int e^{ax}\,dx = \dfrac{e^{ax}}{a} + C$	$\int \csc(x)\cot(x)dx = -\csc(x) + C$

Example 5

Evaluate the following integral

$$\int_0^1 2xe^{-x^2}\,dx$$

Solution

Let $u = -x^2,\ du = -2xdx$

At $x = 0,\ u = -(0)^2 = 0$

At $x = 1,\ u = -(1)^2 = -1$

$$\int_0^1 2xe^{-x^2}\,dx = \int_0^1 (-e^{-x^2})(-2xdx)$$

$$= \int_0^{-1} (-e^u)(du)$$

$$= \left[-e^u\right]_0^{-1}$$

$$= -e^{-1} - (-e^0)$$

=0.6321

Example 6

Evaluate

$$\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x} dx$$

Solution

$$\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x} dx = \int_0^{\pi/4} \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \right) dx$$

$$= \int_0^{\pi/4} (\sec^2 x + \sec x \times \tan x) dx$$

$$= \int_0^{\pi/4} (\sec^2 x) dx + \int_0^{\pi/4} (\sec x)(\tan x) dx$$

$$= [\tan x]_0^{\pi/4} + [\sec x]_0^{\pi/4}$$

$$= (1-0) + (\sqrt{2} - 1)$$

$$= \sqrt{2}$$

Example 7

Evaluate $\int x \sec^2 x \, dx$

Solution

We use the formula

$$\int u dv = uv - \int v du$$

Let $u = x,\ du = dx$, and $dv = \sec^2 x \, dx, v = \tan x$

So the new integral is

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$$

$$= x\tan x + \ln\left|\cos x\right| + C$$

Example 8

Evaluate

$$\int_1^2 x\ln x\,dx$$

Solution

Let $u = \ln x,\ du = \frac{1}{x}dx$ and $dv = x\,dx, v = \frac{x^2}{2}$

Using the formula $\int u\,dv = uv - \int v\,du$, the new integral is

$$\int_1^2 (x)(\ln x)dx = \left[\ln x \times \frac{x^2}{2}\right]_1^2 - \int_1^2 \left(\frac{x^2}{2}\right)\left(\frac{1}{x}dx\right)$$

$$= \left[\ln x \times \frac{x^2}{2}\right]_1^2 - \int_1^2 \frac{x}{2}dx$$

$$= \left[\ln x \times \frac{x^2}{2}\right]_1^2 - \left[\frac{x^2}{4}\right]_1^2$$

$$= \left[\left(\ln 2 \times \frac{2^2}{2}\right) - \left(\ln 1 \times \frac{1^2}{2}\right)\right] - \left[\left(\frac{2^2}{4}\right) - \left(\frac{1^2}{4}\right)\right]$$

$$= \left[(2\ln 2) - \left(\frac{1}{2}\ln 1\right)\right] - \left[\left(\frac{4}{4}\right) - \left(\frac{1}{4}\right)\right]$$

$$= \left[(2\ln 2) - \left(\frac{1}{2} \times 0\right)\right] - \left[1 - \frac{1}{4}\right]$$

$$= 0.6362$$

Example 9

Evaluate

$$\int_0^1 \frac{5x}{(4+x^2)^2}\,dx$$

Solution

We use the formula $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$, by substituting $u = g(x)$, $du = g'(x)dx$ then integrating from $g(a)$ to $g(b)$.

Let

$$u = g(x) = 4 + x^2,$$

so

$g(0) = 4,\ g(1) = 5$, and

$$du = (2x)dx$$

The new integral is

$$\int_0^1 \frac{5x}{(4+x^2)^2}dx = \int_0^1 \frac{1}{(4+x^2)^2} \times \frac{5}{2} \times (2x)dx$$

$$= \frac{5}{2}\int_4^5 \frac{1}{u^2}\,du$$

$$= \frac{5}{2}\left[-\frac{1}{u}\right]_4^5$$

$$= \frac{5}{2}\left[(-\frac{1}{5}) - (-\frac{1}{4})\right]$$

$$= \frac{5}{2} \times \frac{1}{20}$$

= 0.125

Example 10

Evaluate

$$\int_0^4 |2x-1|dx$$

Solution

First, let's analyze the expression $|2x-1|$.

$$|2x-1| = -(2x-1)\,,\ x < \frac{1}{2}$$

$$= (2x-1)\,,\ x \geq \frac{1}{2}$$

$$\int_0^4 |2x-1|dx = \int_0^{1/2} -(2x-1)dx + \int_{1/2}^4 (2x-1)dx$$

$$= \left[-x^2+x\right]_0^{1/2} + \left[x^2-x\right]_{1/2}^4$$

$$= \left[\left(-\frac{1}{4}+\frac{1}{2}\right)-0\right] + \left[(16-4)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]$$

$$= 12.5$$

Example 11

Evaluate

$$\int_{-\infty}^{-2} \frac{2}{x^2-1}dx$$

Solution

$$\int_{-\infty}^{-2} \frac{2}{x^2-1}dx = \int_{-\infty}^{-2} \frac{2}{(x-1)\times(x+1)}dx$$

$$= \int_{-\infty}^{-2} \frac{(x+1)-(x-1)}{(x-1)\times(x+1)} dx$$

$$= \int_{-\infty}^{-2} \frac{x+1}{(x-1)\times(x+1)} - \frac{x-1}{(x-1)\times(x+1)} dx$$

$$= \int_{-\infty}^{-2} \frac{1}{x-1} dx - \int_{-\infty}^{-2} \frac{1}{x+1} dx$$

$$= \lim_{b\to-\infty} \left[\ln|x-1|\right]_b^{-2} - \lim_{b\to-\infty} \left[\ln|x+1|\right]_b^{-2}$$

$$= \lim_{b\to-\infty} \left[\ln\left|\frac{x-1}{x+1}\right| \right]_{-2}^{b}$$

$$= \lim_{b\to-\infty} \left[\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right| \right]$$

$$= \ln(3) - \ln\left(\lim_{b\to-\infty} \left|\frac{b-1}{b+1}\right| \right)$$

$$= \ln(3) - \ln(1)$$

$$= \ln(3)$$

$$= 1.0986$$

Example 12

Graph the function $y = \frac{1}{3}(x^2+2)^{3/2}$, and find the length of the curve from $x = 0$ to $x = 3$.

Solution

We use the equation

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

We have:

$$y = \frac{1}{3}(x^2 + 2)^{3/2}$$

So,

$$\frac{dy}{dx} = \left(\frac{1}{3}\right) \times \left(\frac{3}{2}\right) \times \left(x^2 + 2\right)^{3/2-1} \times (2x)$$

$$= x\sqrt{x^2 + 2}$$

$$L = \int_0^3 \sqrt{1 + \left(x\sqrt{x^2 + 2}\right)^2}\, dx$$

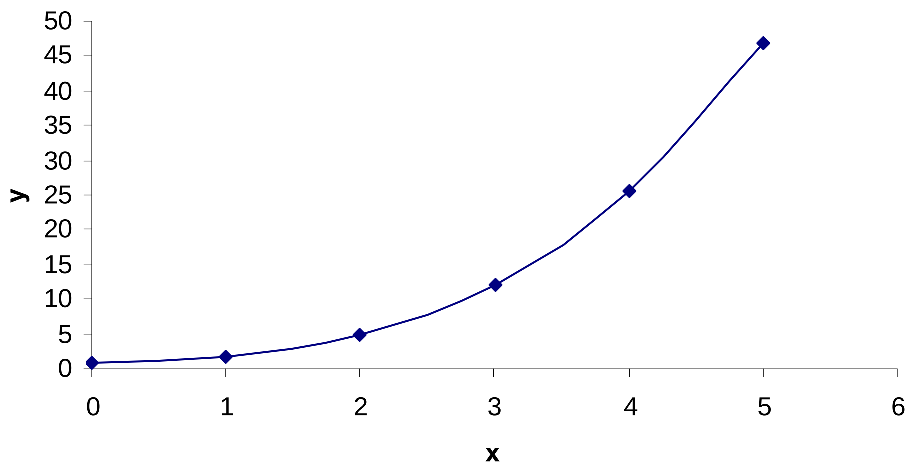


Figure 7 Graph of the function $y = \frac{1}{3}(x^2 + 2)^{3/2}$

$$= \int_0^3 \sqrt{1 + x^2(x^2 + 2)}\,dx$$

$$= \int_0^3 \sqrt{1 + x^4 + 2x^2}\,dx$$

$$= \int_0^3 \sqrt{(x^2 + 1)^2}\,dx$$

$$= \int_0^3 (x^2 + 1)\,dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^3$$

$$= 12$$

Example 13

Find the area of the shaded region given in Figure 8.

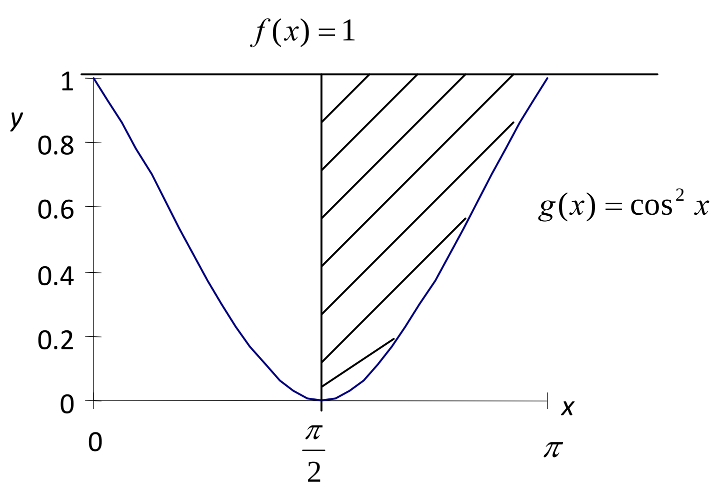


Figure 8 Graph of the function $\cos^2 x$.

Solution

For the sketch given,

$a=\dfrac{\pi}{2}$, $b=\pi$, and

$$f(x)-g(x)=1-\cos^2 x=\sin^2 x$$

$$A=\int_{\pi/2}^{\pi}\sin^2(x)\,dx$$

$$=\int_{\pi/2}^{\pi}\frac{1-\cos 2x}{2}\,dx$$

$$=\int_{\pi/2}^{\pi}\left[\frac{1}{2}-\frac{\cos 2x}{2}\right]dx$$

$$=\left[\frac{x}{2}-\frac{\sin 2x}{4}\right]_{\pi/2}^{\pi}$$

$$=\left[\left(\frac{\pi}{2}-\frac{\sin(2\pi)}{4}\right)-\left(\frac{\pi}{4}-\frac{\sin 2\left(\frac{\pi}{2}\right)}{4}\right)\right]$$

$$=\left[\left(\frac{\pi}{2}-0\right)-\left(\frac{\pi}{4}-0\right)\right]$$

$$=\frac{\pi}{4}$$

Example 14

Find the volume of the solid generated by revolving the shaded region in Figure 9 about the *y*-axis.

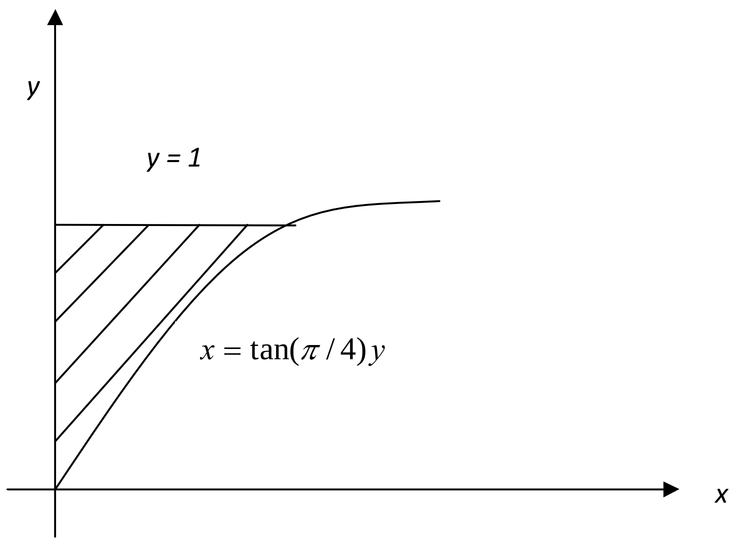


Figure 9 Volume generated by revolving shaded region.

Solution

We use the formula $V = \int_a^b \pi(radius)^2 dy$

Let

$$u = \frac{\pi}{4}y,\ du = \frac{\pi}{4}dy\,.$$

Therefore, at $y = 0,\ u = 0$

$$y = 1,\ u = \frac{\pi}{4}$$

$$V = \int_0^1 \pi[R(y)]^2 dy$$

$$= \pi\int_0^1 \left[\tan\left(\frac{\pi}{4}y\right)\right]^2 dy$$

$$= \pi \times \frac{4}{\pi} \int_0^1 \left[\tan\left(\frac{\pi}{4} y \right) \right]^2 \frac{\pi}{4} dy$$

$$= 4 \int_0^{\pi/4} (\tan u)^2 \, du \quad \text{(Choosing } u = \frac{\pi}{4} y \text{)}$$

$$= 4 \int_0^{\pi/4} \left(-1 + \sec^2 u \right) du$$

$$= 4 \left[-u + \tan u \right]_0^{\pi/4}$$

$$= 4 \left[\left(-\frac{\pi}{4} + \tan\frac{\pi}{4} \right) - \left(0 + \tan 0 \right) \right]$$

$$= 4 \left[\left(-\frac{\pi}{4} + 1 \right) - \left(0 + 0 \right) \right]$$

$$= 0.8584$$

Partial Fractions:-

Partial Fractions provides a way to integrate all rational functions.

Rational functions= $\frac{P(x)}{Q(x)}$ when P and Q are polynomials

This is the technique to find $\int \frac{P(x)}{Q(x)} dx$

Rule 1: The degree of the numerator must be less than the degree of the denominator. If this is not the case we first must divide the numerator into the denominator.

Step 1: If Q has a quadratic factor $ax^2 + bx + c$ which corresponds to a complex root of order k, then the partial fraction expansion of $\frac{P}{Q}$ contains a term of the form

$$\frac{B_1x + C_1}{(ax^2 + bx + c)} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \ldots + \frac{B_k + C_k}{(ax^2 + bx + c)^k}$$

Where B_1, B_2, … B_k and C_1 , C_2 , …,C_k are unknown constants.

Step 2: Set the sum of the terms of equal to the partial fraction expansion

Example: $\frac{1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}$

Step 3: When then multiply both sides by Q to get some expression that is equal to P

Example: 1= A(x-5) + B(x-2)

1= (A+B)x-5A-2B

Step 4: Use the theory that 2 polynomials are equal if and only if the corresponding coefficients are equal

Example: 5A-2B=1 and A+B=0

Step 5: Solve for A, B, and C

Example: A= -1/3 B= 1/3

Step 6: Express integral of $\frac{P}{Q}$ as the sum of the integrals of the terms of partial fraction expansion.

Example: $\int \frac{1}{(x-2)(x-5)}dx = \int \frac{\frac{-1}{3}}{(x-2)}dx + \int \frac{\frac{1}{3}}{(x-5)}dx$

= $\frac{-1}{3}\ln|x-2| + \frac{1}{3}\ln|x-5| + C$

Example 2:

Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}dx$

$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x+1+\frac{4x}{x^3 - x^2 - x + 1}$ Note: long division

$\frac{4x}{(x-1)^2(x+1)}$ Note: Factor Q(x)= $x^3 - x^2 - x + 1$

$\frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$ Note: Partial fraction decomposition since $(x-1)^2$'s factor is linear. There is a constant on top for the and power and first power

4x= A(x-1)(x+1) + B(x+1) + C$(x-1)^2$

Note: multiply by Least common denominator

$(x-1)^2$ (x+1)

= $(A+C)x^2 + (B-2C)x + (-A+B+C)$

$A+C = 0$

$B-2C = 4$

$-A+B+C = 0$ Note: Equate equations

$A=1$ $B=2$ $C=-1$ Note: Solve for coefficients

$$\int (x+1)dx + \int \frac{1}{x-1}dx + \int \frac{2}{(x-1)^2}dx - \int \frac{1}{x+1}dx$$

$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + C$$

Example 3:

Find $\int \frac{2x^2 - x + 4}{x^3 + 4x}dx$

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$ Note: x^2+4 is quadratic

$2x^2 - x + 4 = A(x^2+4) + (Bx+C)x$ Note: multiplying $x(x^2+4)$

$= (A+B)x^2 + Cx + 4A$

A+B=2 C=-1 4A=4 Note: Equating coefficients

A=1 B=1 C=-1

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx = \int \frac{1}{x} dx + \int \frac{x-1}{x^2+4} dx = \int \frac{1}{x} dx + \int \frac{x}{x^2+4} dx - \int \frac{1}{x^2+4} dx$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

INTEGRATION BY PART

This is a method to evaluate integrals that cannot be evaluated by eye or by u-substitution. It is usually applied to expressions with varied functions within each other, or multiplied by each other. A good rule is: if the expression has a chain of functions (f(g(x)) or if the expression has a product of functions x(f(x)), integration by parts will be necessary. Here are some examples of problems that would be solved with integration by parts:

$x \cdot \sec^2(x)$ $\qquad$ $x^3 \ln(x)$ $\qquad$ $\theta^2 \sin\theta$

Let's start with:

$$x^3 \ln(x)$$

In integration by parts, you separate the expression into two parts: u, and $\partial(v)$.

The u should be easy to differentiate, and the $d(v)$ should be easy to integrate.

Once you have chosen a u and a $d(v)$, set up a chart like this:

u = ln(x) $\quad d(v) = \frac{\ln(x)\cdot x^4}{4} - \frac{x^4}{16}$

$d(u) = x^{-1}$ $\quad$ v = $\frac{x^4}{4}$

Now, the formula to solve this is:

$$uv - \int \left(d(u)v\right)$$

so here, the equation to solve is:

$$\frac{\ln(x)\cdot x^4}{4} - \int \left(x^{-1}\cdot\frac{x^4}{4}\right) d(x)$$

which simplifies to:

$$\frac{\ln(x)\cdot x^4}{4} - \int \left(\frac{x^3}{4}\right) d(x)$$

and solve the integral to get:

$$\frac{\ln(x)\cdot x^4}{4} - \frac{x^4}{16}$$

Unfortunately, it is not always so simple. Sometimes, you must use u-substitution, or even integration by parts again within the solution. Take, for example:

$$\theta^2 \sin\theta$$

To solve this, you would set up a chart again.

$$\theta^2 \sin\theta$$

u = θ^2	$d(v)$ = $\sin\theta$
$d(u)$ = 2θ	v = $-\cos\theta$

With this chart, you can set up the solution using the $uv - \int (d(u)v)$ formula:

$$= -\theta^2 \cos\theta + \int (2\theta\cos\theta)\, d(\theta)$$

But the second part of this, $\int (2\theta\cos\theta)\, d(\theta)$ cannot be solved by eye. You must set up a second chart:

$$\int (2\theta\cos\theta)\, d(\theta)$$

u = 2θ	$d(v)$ = $\cos\theta$
$d(u)$ = 2	v = $\sin\theta$

This gives us:

$$= 2\theta\sin\theta + \int (2\sin\theta)\, d(\theta)$$

Which can be simplified to:

$$= 2\theta\sin\theta + 2\cos\theta$$

Now, you can substitute it into the original solution, in the place of $\int (2\theta\cos\theta)\, d(\theta)$, giving you:

$$= -\theta^2 \cos\theta + 2\theta\sin\theta + 2\cos\theta$$

Trigonometric Integrals(REDUCTION FORMULA)

I. Integrating Powers of the Sine and Cosine Functions

A. Useful trigonometric identities

1. $\sin^2 x + \cos^2 x = 1$

2. $\sin 2x = 2\sin x \cos x$

3. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

4. $\sin^2 x = \dfrac{1-\cos 2x}{2}$

5. $\cos^2 x = \dfrac{1+\cos 2x}{2}$

6. $\sin x \cos y = \dfrac{1}{2}[\sin(x-y) + \sin(x+y)]$

7. $\sin x \sin y = \dfrac{1}{2}[\cos(x-y) - \cos(x+y)]$

8. $\cos x \cos y = \dfrac{1}{2}[\cos(x-y) + \cos(x+y)]$

B. Reduction formulas

1. $\int \sin^n x \, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, dx$

2. $\int \cos^n x \, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, dx$

C. Examples

1. Find $\int \sin^2 x \, dx$.

Method 1(Integration by parts): $\int \sin^2 x \, dx = \int \sin x \, (\sin x \, dx)$. Let

$u = \sin x$ and $dv = \sin x \, dx \Rightarrow du = \cos x \, dx$ and $v = \int \sin x \, dx =$

$-\cos x$. Thus, $\int \sin^2 x \, dx = (\sin x)(-\cos x) + \int \cos^2 x \, dx = -\sin x \cos x +$

$\int (1 - \sin^2 x) \, dx = -\sin x \cos x + \int 1 dx - \int \sin^2 x \, dx = -\sin x \cos x + x -$

$\int \sin^2 x \, dx \Rightarrow 2\int \sin^2 x \, dx = -\sin x \cos x + x \Rightarrow \int \sin^2 x \, dx =$

$-\frac{1}{2}\sin x \cos x + \frac{1}{2}x + C$.

Method 2(Trig identity): $\int \sin^2 x\ dx = \frac{1}{2}\int (1-\cos 2x)\ dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$.

Method 3(Reduction formula): $\int \sin^2 x\ dx = -\frac{1}{2}\sin x\cos x + \frac{1}{2}\int 1dx =$

$-\frac{1}{2}\sin x\cos x + \frac{1}{2}x + C$.

2. Find $\int \cos^3 x\ dx$.

Use the reduction formula: $\int \cos^3 x\ dx = \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\int \cos x\, dx =$

$\frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin x + C = \frac{1}{3}\sin x(1-\sin^2 x) + \frac{2}{3}\sin x + C =$

$\sin x - \frac{1}{3}\sin^3 x + C$.

3. Find $\int \sin^3 x\cos^2 x\ dx$.

$\int \sin^3 x\cos^2 x\ dx = \int \sin^2 x\sin x\cos^2 x\ dx = \int (1-\cos^2 x)\cos^2 x\sin x dx =$

$\int (\cos^2 x - \cos^4 x)(\sin x\ dx)$. Let $u = \cos x \Rightarrow du = -\sin x\, dx$. Thus,

$$\int(\cos^2 x-\cos^4 x)(\sin x\ dx)=-\int(u^2-u^4)\ du=-\frac{1}{3}u^3+\frac{1}{5}u^5+C=$$

$$-\frac{1}{3}\cos^3 x+\frac{1}{5}\cos^5 x+C.$$

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4. Find $\int \sin^2 x\cos^2 x\ dx$.

$$\int \sin^2 x\cos^2 x\ dx=\int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx=\frac{1}{4}\int(1-\cos^2 2x)\,dx=$$

$$\frac{1}{4}\int \sin^2 2x\,dx=\frac{1}{4}\int\left(\frac{1-\cos 4x}{2}\right)dx=\frac{1}{8}\int 1dx-\frac{1}{8}\int \cos 4x\ dx=$$

$$\frac{1}{8}x-\frac{1}{32}\sin 4x+C.$$

5. Find $\int \sin 4x\cos 3x\ dx$.

Method 1(Integration by parts): Let $u=\sin 4x$ and $dv=\cos 3x\ dx \Rightarrow$ *du* =

$4\cos 4x\ dx$ and $v=\frac{1}{3}\sin 3x$. Thus, $\int \sin 4x\cos 3x\ dx=$

$(\sin 4x)\left(\frac{1}{3}\sin 3x\right)-\frac{4}{3}\int \cos 4x\sin 3x\ dx=\frac{1}{3}\sin 4x\sin 3x-$

$\frac{4}{3}\int \cos 4x\sin 3x\ dx$. Find $\int \cos 4x\sin 3x\ dx$. Let $u=\cos 4x$ and *dv* =

$\sin 3x\ dx \Rightarrow du = -4\sin 4x\ dx$ and $v = -\frac{1}{3}\cos 3x$. Thus,

$\int \cos 4x \sin 3x\ dx = -\frac{1}{3}\cos 4x \cos 3x - \frac{4}{3}\int \sin 4x \cos 3x\ dx$. Returning to

the original integral, $\int \sin 4x \cos 3x\ dx$ = $\frac{1}{3}\sin 4x \sin 3x -$

$\frac{4}{3}\left\{-\frac{1}{3}\cos 4x \cos 3x - \frac{4}{3}\int \sin 4x \cos 3x\ dx\right\} = \frac{1}{3}\sin 4x \sin 3x +$

$\frac{4}{9}\cos 4x \cos 3x + \frac{16}{9}\int \sin 4x \cos 3x\ dx \Rightarrow -\frac{7}{9}\int \sin 4x \cos 3x\ dx =$

$\frac{1}{3}\sin 4x \sin 3x + \frac{4}{9}\cos 4x \cos 3x \Rightarrow \int \sin 4x \cos 3x\ dx$ =

$-\frac{3}{7}\sin 4x \sin 3x - \frac{4}{7}\cos 4x \cos 3x + C$.

Method 2(Trig identity): $\int \sin 4x \cos 3x\ dx = \frac{1}{2}\int (\sin x + \sin 7x)\ dx =$

$-\frac{1}{2}\cos x - \frac{1}{14}\cos 7x + C$.

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II. Integrating Powers of the Tangent and Secant Functions

A. Useful trigonometric identity: $\tan^2 x + 1 = \sec^2 x$

B. Useful integrals

1. $\int \sec x \tan x \; dx = \sec x + C$

2. $\int \sec^2 x \, dx = \tan x + C$

3. $\int \tan x \; dx = \ln\left|\sec x\right| + C = -\ln\left|\cos x\right| + C$

4. $\int \sec x \, dx = \ln\left|\sec x + \tan x\right| + C$

C. Reduction formulas

1. $\int \sec^n x \, dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2} x \, dx$

2. $\int \tan^n x \, dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$

D. Examples

1. Find $\int \tan^2 x \, dx$.

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \int \sec^2 x\,dx - \int 1dx = \tan x - x + C\,.$$

2. Find $\int \tan^3 x dx$.

$$\int \tan^3 x dx = \frac{\tan^2 x}{2} - \int \tan x\,dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + C\,.$$

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3. Find $\int \sec^3 x dx$.

$$\int \sec^3 x\,dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\int \sec x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C\,.$$

4. Find $\int \tan x \sec^2 x\,dx$.

Let $u = \tan x \Rightarrow du = \sec^2 xdx \Rightarrow \int \tan x \sec^2 x\,dx = \int udu = \frac{1}{2}u^2 + C =$

$\frac{1}{2}\tan^2 x + C\,.$

5. Find $\int \tan x \sec^4 x\, dx$.

$$\int \tan x \sec^4 x\, dx = \int \tan x \sec^2 x \sec^2 x\, dx = \int \tan x(1+\tan^2 x)\sec^2 x\, dx =$$

$\int \tan x \sec^2 x\, dx + \int \tan^3 x \sec^2 dx$. Let $u = \tan x \Rightarrow du = \sec^2 x\, dx$. Thus,

$$\int \tan x \sec^4 x\, dx = \int u du + \int u^3 du = \frac{1}{2}u^2 + \frac{1}{4}u^4 + C = \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C.$$

6. Find $\int \tan x \sec^3 x\, dx$.

$\int \tan x \sec^3 x\, dx = \int \sec^2 x(\sec x \tan x\, dx)$. Let $u = \sec x \Rightarrow du = \sec x \tan x dx$.

Thus, $\int \tan x \sec^3 x\, dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C$.

7. Find $\int \tan^2 x \sec^3 x\, dx$.

$\int \tan^2 x \sec^3 x\, dx = \int (\sec^2 x - 1)\sec^3 x\, dx = \int \sec^5 x\, dx - \int \sec^3 x\, dx$. Using the reduction formula, $\int \sec^5 x\, dx = \frac{1}{4}\sec^3 \tan x + \frac{3}{4}\int \sec^3 x\, dx$. Thus,

$$\int \tan^2 x \sec^3 x\,dx = \int \sec^5 x\,dx - \int \sec^3 x\,dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x dx -$$

$$\int \sec^3 x\,dx = \frac{1}{4}\sec^3 x \tan x - \frac{1}{4}\int \sec^3 x\,dx = \frac{1}{4}\sec^3 x \tan x - \frac{1}{8}\sec x \tan x -$$

$$\frac{1}{8}\ln\left|\sec x + \tan x\right| + C.$$

8. Find $\int \sqrt{\tan x}\ \sec^4 x\ dx$.

$$\int \sqrt{\tan x}\ \sec^4 x\ dx = \int \sqrt{\tan x} \sec^2 x \sec^2 x\,dx = \int \sqrt{\tan x}\,(1+\tan^2 x)\sec^2 x dx.$$

Let $u = \tan x \Rightarrow du = \sec^2 x\,dx \Rightarrow \int \sqrt{\tan x}\ \sec^4 x\ dx = \int \sqrt{\tan x} \sec^2 x\,dx +$

$$\int \sqrt{\tan x} \tan^2 x \sec^2 x\ dx = \int u^{1/2} du + \int u^{5/2} du = \frac{2}{3}u^{3/2} + \frac{2}{7}u^{7/2} + C =$$

$$\frac{2}{3}(\tan x)^{3/2} + \frac{2}{7}(\tan x)^{7/2} + C.$$

9. Find $\int \sqrt{\sec x} \tan x\,dx$.

Let $u = \sqrt{\sec x} \Rightarrow u^2 = \sec x \Rightarrow 2udu = \sec x \tan x dx = u^2 \tan x dx \Rightarrow$

$\tan x\,dx = \dfrac{2udu}{u^2} = \dfrac{2}{u}du$. Thus, $\int \sqrt{\sec x} \tan x\,dx = \int u\left(\dfrac{2}{u}du\right) = 2\int 1du =$

$$2u + C = 2\sqrt{\sec x} + C.$$

Practice Sheet forTrigonometric Integrals

(1) Prove the reduction formula: $\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

(2) Prove the reduction formula: $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

(3) Prove the reduction formula: $\int \sec^n x\,dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$

(4) Prove the reduction formula: $\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

(5) $\int_0^{\pi/4} \tan^3(3x)\ dx =$

(6) $\int_0^{\pi/4} \cos^2(2x)\ dx =$

(7) $\int_0^{\pi/8} \sin(5x)\cos(3x)\ dx =$

(8) $\int \tan^3 x \sec^3 x\ dx =$

(9) $\int \sqrt{\sin x}\cos^3 x \ dx =$

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(10) $\int \cos^3 x \sin^2 x \ dx =$

(11) $\int_0^{\pi/2} \frac{\sin^3 x}{\sqrt{\cos x}} \ dx =$

(12) $\int \sin^2 x \cos^2 x dx =$

(13) $\int \tan^5 x \sec x dx =$

Solution Key for Trigonometric Integrals

(1) $\int \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx$. Use integration by parts with $u = \sin^{n-1} x$ and

$dv = \sin x\,dx \Rightarrow du = (n-1)\sin^{n-2} x \cos x\,dx$ and $v = \int \sin x\,dx = -\cos x \Rightarrow$

$$\int \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx =$$

$$-\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\left(1-\sin^2 x\right)dx = -\sin^{n-1} x \cos x +$$

$$(n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx \Rightarrow n\int \sin^n x\,dx = -\sin^{n-1} x \cos x +$$

$$(n-1)\int \sin^{n-2} x\,dx \Rightarrow \int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

(2) $\int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx$. Use integration by parts with $u = \cos^{n-1} x$ and

$dv = \cos x\,dx \Rightarrow du = (n-1)\cos^{n-2} x(-\sin x)\,dx$ and $v = \int \cos x\,dx = \sin x \Rightarrow$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx =$$

$$\cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\left(1-\cos^2 x\right)dx = \cos^{n-1} x \sin x +$$

$$(n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx \Rightarrow n\int \cos^n x\,dx = \cos^{n-1} x \sin x +$$

$$(n-1)\int \cos^{n-2} x\,dx \Rightarrow \int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx\,.$$

(3) $\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$. Use integration by parts with $u = \sec^{n-2} x$ and

$dv = \sec^2 x\,dx \Rightarrow du = (n-2)\sec^{n-3} x\,(\sec x \tan x\,dx)$ and $v = \int \sec^2 x\,dx = \tan x \Rightarrow$

$$\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx =$$

$$\sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\left(\sec^2 x - 1\right)dx = \sec^{n-2} x \tan x - (n-2)\int \sec^n x dx +$$

$$(n-2)\int \sec^{n-2} x\,dx \Rightarrow (n-1)\int \sec^n x\,dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\,dx \Rightarrow$$

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx\,.$$

(4) $\int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx = \int \tan^{n-2} x\left(\sec^2 x - 1\right)dx = \int \tan^{n-2} x \sec^2 x dx -$

$$\int \tan^{n-2} x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx\,.$$

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(5) Let $u = 3x \Rightarrow du = 3\,dx \Rightarrow \int \tan^3(3x)\,dx = \frac{1}{3}\int \tan^3(3x)\;3dx = \frac{1}{3}\int \tan^3 u\,du$. Use

reduction formula #4 above to get $\frac{1}{3}\int \tan^3 u\,du = \frac{1}{3}\left(\frac{\tan^2 u}{2}\right) - \frac{1}{3}\int \tan u\,du =$

$\frac{1}{6}\tan^2 u - \frac{1}{3}\ln|\sec u| \Rightarrow \int_0^{\pi/4} \tan^3(3x)\;dx = \left\{\frac{1}{6}\tan^2(3x) - \frac{1}{3}\ln|\sec(3x)|\right\}_0^{\pi/4} =$

$\left\{\frac{1}{6}\tan^2\left(\frac{3\pi}{4}\right) - \frac{1}{3}\ln\left|\sec\left(\frac{3\pi}{4}\right)\right|\right\} - \left\{\frac{1}{6}\tan^2(0) - \frac{1}{3}\ln|\sec(0)|\right\} = \frac{1}{6}(-1)^2 - \frac{1}{3}\ln\left|-\sqrt{2}\right| -$

$$\frac{1}{6}(0)^2+\frac{1}{3}\ln 1=\frac{1}{6}-\frac{1}{3}\ln\left(\sqrt{2}\right).$$

(6) Use the trigonometric identity $\cos^2\Delta=\dfrac{1+\cos 2\Delta}{2}$ to get $\int\cos^2(2x)\,dx=$

$$\int\frac{1+\cos(4x)}{2}dx=\frac{1}{2}\int 1dx+\frac{1}{2}\int\cos(4x)\,dx=\frac{1}{2}x+\frac{1}{8}\sin(4x)\Rightarrow\int_0^{\pi/4}\cos^2(2x)\ dx=$$

$$\left\{\frac{1}{2}\left(\frac{\pi}{4}\right)+\frac{1}{8}\sin\pi\right\}-\left\{\frac{1}{2}(0)+\frac{1}{8}\sin(0)=\frac{\pi}{8}\right\}.$$

(7) Use the trigonometric identity $\sin x\cos y=\dfrac{1}{2}[\sin(x-y)+\sin(x+y)]$ to get

$$\int\sin(5x)\cos(3x)\,dx=\frac{1}{2}\int\sin(2x)\,dx+\frac{1}{2}\int\sin(8x)\,dx=-\frac{1}{4}\cos(2x)-\frac{1}{16}\cos(8x)\Rightarrow$$

$$\int_0^{\pi/8}\sin(5x)\cos(3x)\,dx=\left\{-\frac{1}{4}\cos\left(\frac{\pi}{4}\right)-\frac{1}{16}\cos(\pi)\right\}-\left\{-\frac{1}{4}\cos 0-\frac{1}{16}\cos 0\right\}=$$

$$-\frac{1}{4}\left(\frac{\sqrt{2}}{2}\right)+\frac{1}{16}+\frac{1}{4}+\frac{1}{16}=\frac{3-\sqrt{2}}{8}$$

(8) $\int \tan^3 x \sec^3 x \; dx = \int \tan^2 x \sec^2 x (\sec x \tan x \, dx) =$

$\int (\sec^2 x - 1)\sec^2 x (\sec x \tan x \, dx) = \int \sec^4 x (\sec x \tan x \, dx) -$

$\int \sec^2 x (\sec x \tan x \, dx) = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.$

(9) $\int \sqrt{\sin x} \cos^3 x \; dx = \int \sqrt{\sin x} \left(\cos^2 x\right)(\cos x \, dx) = \int (\sin x)^{1/2}\left(1 - \sin^2 x\right)\cos x dx =$

$\int (\sin x)^{1/2} \cos x \, dx - \int (\sin x)^{5/2} \cos x \, dx = \frac{2}{3}(\sin x)^{3/2} - \frac{2}{7}(\sin x)^{7/2} + C.$

(10) $\int \cos^3 x \sin^2 x \; dx = \int \cos^2 x \sin^2 x (\cos x \, dx) = \int \left(1 - \sin^2 x\right)\left(\sin^2 x\right)\cos x \, dx =$

$\int \sin^2 x (\cos x \, dx) - \int \sin^4 x (\cos x \, dx) = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C.$

(11) $\int \frac{\sin^3 x}{\sqrt{\cos x}} \, dx = \int (\cos x)^{-1/2} \sin^2 x (\sin x \, dx = \int (\cos x)^{-1/2}\left(1 - \cos^2 x\right)\sin x \, dx =$

$\int (\cos x)^{-1/2}(\sin x \, dx) - \int (\cos x)^{3/2}(\sin x \, dx) = -2(\cos x)^{1/2} + \frac{2}{5}(\cos x)^{5/2} \Rightarrow$

$$\int_0^{\pi/2} \frac{\sin^3 x}{\sqrt{\cos x}}\, dx = \left\{-2\cos\left(\frac{\pi}{2}\right)+\frac{2}{5}\left(\cos\left(\frac{\pi}{2}\right)\right)^{5/2}\right\}-\left\{-2\cos 0+\frac{2}{5}(\cos 0)^{5/2}\right\}=\frac{8}{5}.$$

(12) Use the trigonometric identities $\cos^2\Delta = \dfrac{1+\cos 2\Delta}{2}$ and $\sin^2\Delta = \dfrac{1-\cos 2\Delta}{2}$.

$$\int \sin^2 x\cos^2 x dx = \int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{4}\int\left(1-\cos^2 2x\right)dx =$$

11

$$\frac{1}{4}\int 1dx - \frac{1}{4}\int \cos^2 2x\ dx = \frac{1}{4}x - \frac{1}{4}\int\left(\frac{1+\cos 4x}{2}\right)dx = \frac{1}{4}x - \frac{1}{8}\int 1dx -$$

$$\frac{1}{8}\int \cos 4x\, dx = \frac{1}{4}x - \frac{1}{8}x - \frac{1}{32}\sin 4x + C = \frac{1}{8}x - \frac{1}{32}\sin 4x + C.$$

(13) $\displaystyle\int \tan^5 x\sec x dx = \int \tan^4 x\tan x\sec x\, dx = \int\left(\tan^2 x\right)^2 \tan x\sec x\, dx =$

$$\int\left(\sec^2 x - 1\right)^2 \sec x\tan x\, dx = \int\left(\sec^4 x - 2\sec^2 x + 1\right)\sec x\tan x\, dx =$$

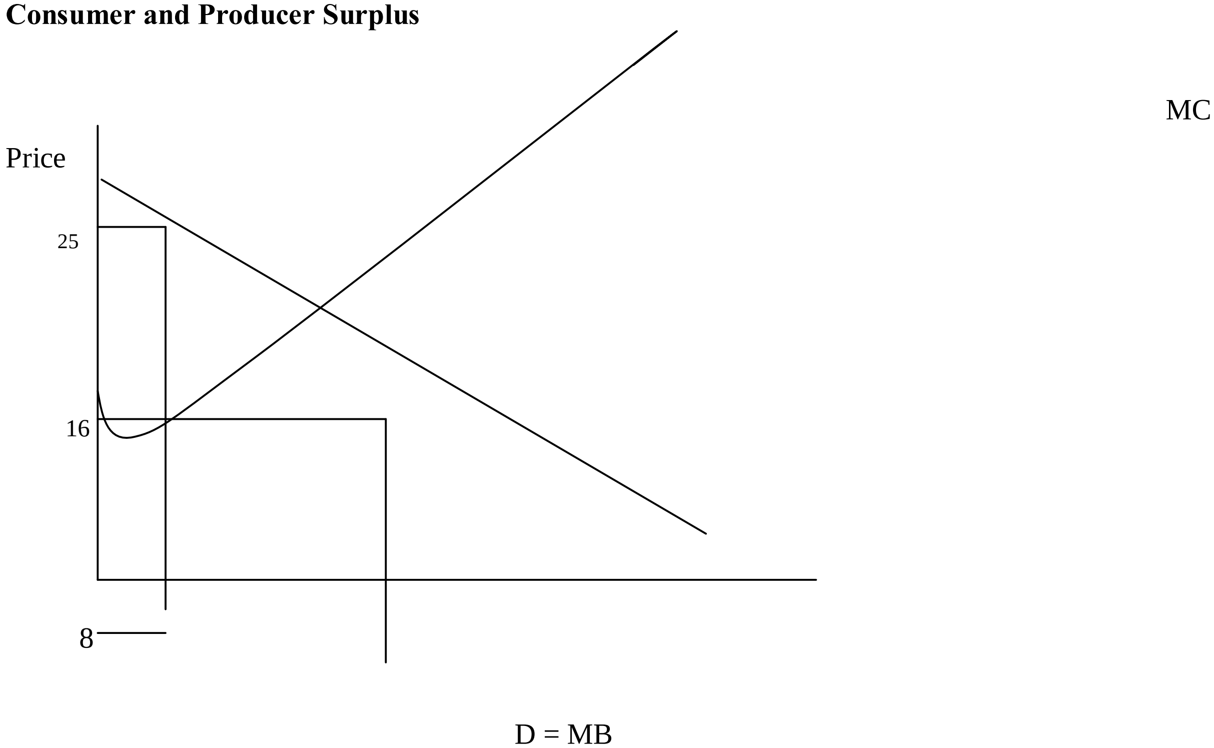
$$\int \sec^4 x(\sec x \tan x\, dx) - 2\int \sec^2 x(\sec x \tan x\, dx) + \int \sec x \tan x\, dx =$$

$$\frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C\,.$$

Consumer Surplus, Producer Surplus

Consumer Surplus is defined as the difference between the price a customer willing to pay for a product and the price that he actually ends up paying. When a consumer gets to purchase a good at a lower price than the price he is willing to pay, he gets more benefits creating a consumer surplus. As an example, for a necessity like food consumer would be willing to pay a higher price as it is a necessity. But at normal market conditions consumer can obtain food at a relatively lower price than what he is willing to pay and it creates a consumer surplus. When the utility (satisfaction) of a good falls the consumer surplus reduces as the consumer will not be willing to pay higher price. The consumer surplus can be visually represented as follows:

Consumer and Producer Surplus



1

6

Quantity

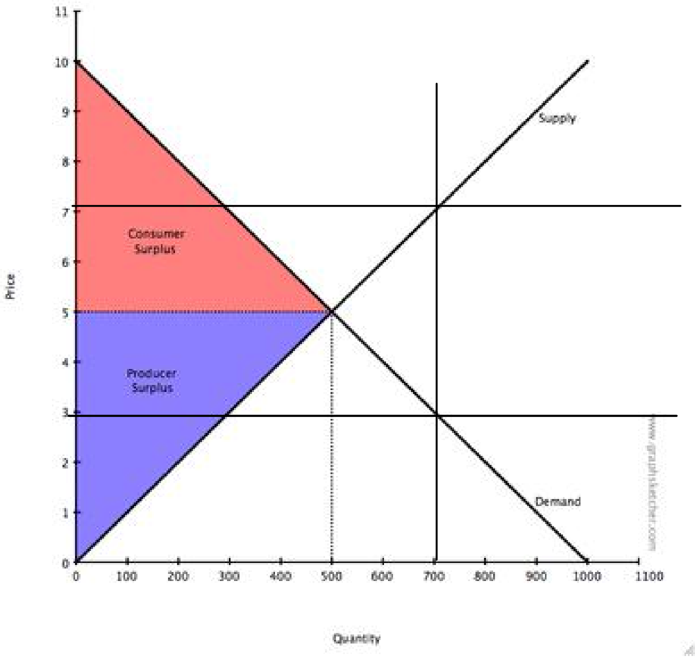
Imagine rummaging through a rack of clothes at TJ Maxx. You come across a shirt, and immediately a price pops into your head. You think to yourself, I would be willing to pay $25 for this shirt. That means, $25 is your marginal benefit (or demand ï the willingness to pay). Then you look at the price and see a tag of $16. You are happy!! You have a consumer surplus of $9.

At the other end of the transaction, TJ Maxx corporate office bought the shirt you are looking at on closeout. They estimate that it costs them $8 to put that shirt on the rack. When you buy the shirt for $16 they are happy too. They have a producer surplus of $8.

In this example, the two economic agents (the store and the customer) should make an exchange.

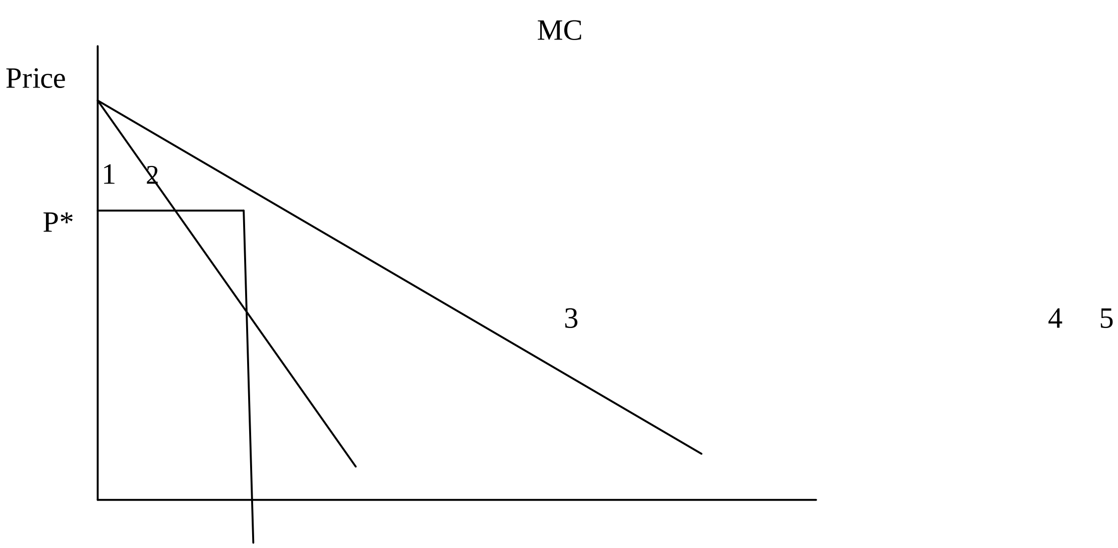
Consumer surplus will exist as long as the marginal benefit to the consumer is greater than or equal to the price the consumer must pay. It is the area between D and P

Produce surplus will exist as long as the price is greater than or equal to the marginal cost of producing the good or service. It is the area between P and MC



In the perfectly competitive model, exchanges will be made all the way over to the interception of demand and supply (equilibrium). After that point, for example in the graph above, when the quantity is 700 the marginal benefit (the amount someone is willing to pay) is about $3 and the marginal cost of producing the good is about $7. In this case, it does not make sense for the two economic agents to make an exchange. To the 700th customer the marginal benefit of the good is only $3. The company's cost to get the good to the 700th customer is $7. Therefore, it is inefficient to make an exchange. This is true at any quantity above 500 in the picture above.

Imperfect Competition Product Market



MR

D

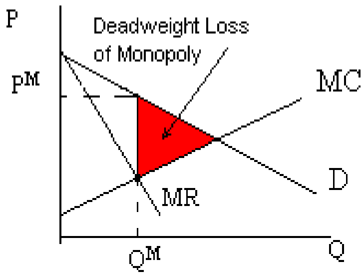
Q*

Quantity

In an imperfect product market the price and quantity are determined from the intersection of MR and MC. (Go down to find Q* and up to the demand curve to find P*). The profit maximizing price is denoted by P* and the profit maximizing quantity is denoted by Q*. Exchanges are made up to the quantity Q*. The area of consumer surplus is the space between the demand curve (also known as the marginal benefit line) and the price. Therefore, the areas of consumer surplus are labeled 1 and 2. The area of producer surplus is the space between price and marginal cost (areas 3 and 4).

There is no surplus after Q* because there are no exchanges made even though the demand (marginal benefit) is greater than the marginal cost. In the area where D > MC there should be an exchange made but there is not (this is shown as area 5 above). This phenomena when D is greater MC but there is no exchange made is known as a dead weight loss. Dead weight losses occur in imperfect competition, or when there are taxes, or when there are price ceilings or floors.

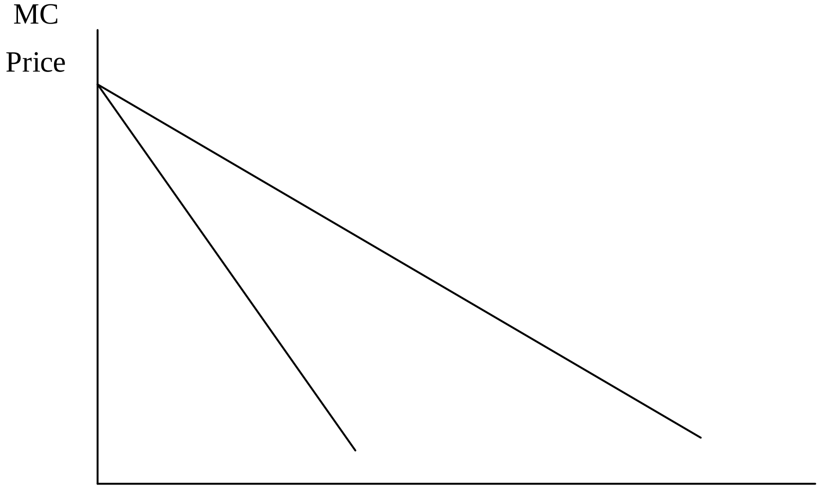
This provides a bit more information on the topic.



Dead weight loss occurs when people who would have more marginal benefit than marginal cost are not buying the product. The company does not want to lower their price to all of their customers because they would lose profits.

Taking advantage of the dead weight loss.

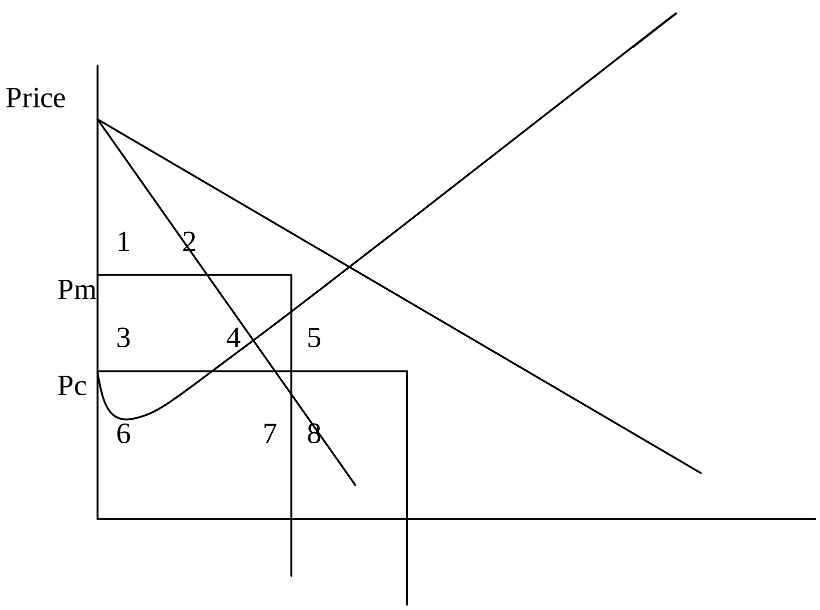
Recently, we have seen a lot of companies recognize the dead weight loss as a potential area of additional profit. From the companyôperspective, these are customers who would buy the product if the price was a little lower. They are on the margin. The company wants these people to be customers but they know they canôt lower their price. So what they do is lower their price for new customers only. Have you ever seen a company do that? Now you know why. When incentives are offered to new customers only, the dead weight loss gets smaller and surpluses increase.



MR

D

MC



MR D

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